Analysis prior with redundant dictionaries for Compressed Sensing

Kévin Degraux

After the article

E. J. Candès, Y. C. Eldar, D. Needell, and P. Randall,

"Compressed sensing with coherent and redundant dictionaries,"

Appl. Comput. Harmon. Anal., vol. 31, no. 1, pp. 59–73, Jul. 2011.

Compressed sensing

$$y \quad = \quad \Phi \quad\quad x$$

Compressed sensing

$$y = \Phi \quad x + n$$

Compressed sensing

$$y = \Phi x + n$$

Meaningful information

$$N$$

Compressed sensing

$$y = \Phi$$

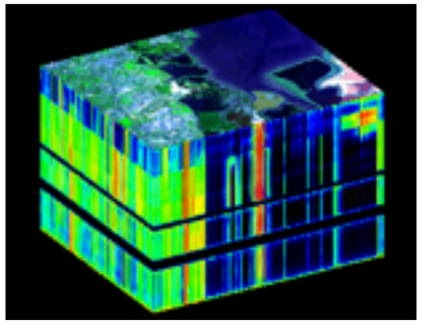
$+n$

Meaningful information

N

Compressed sensing

$y \quad = \quad \Phi$

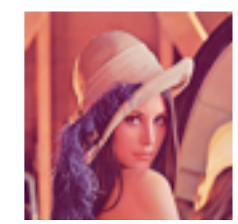


$+n$

Meaningful information

N

Compressed sensing

$y \quad = \quad \Phi$  $+n$

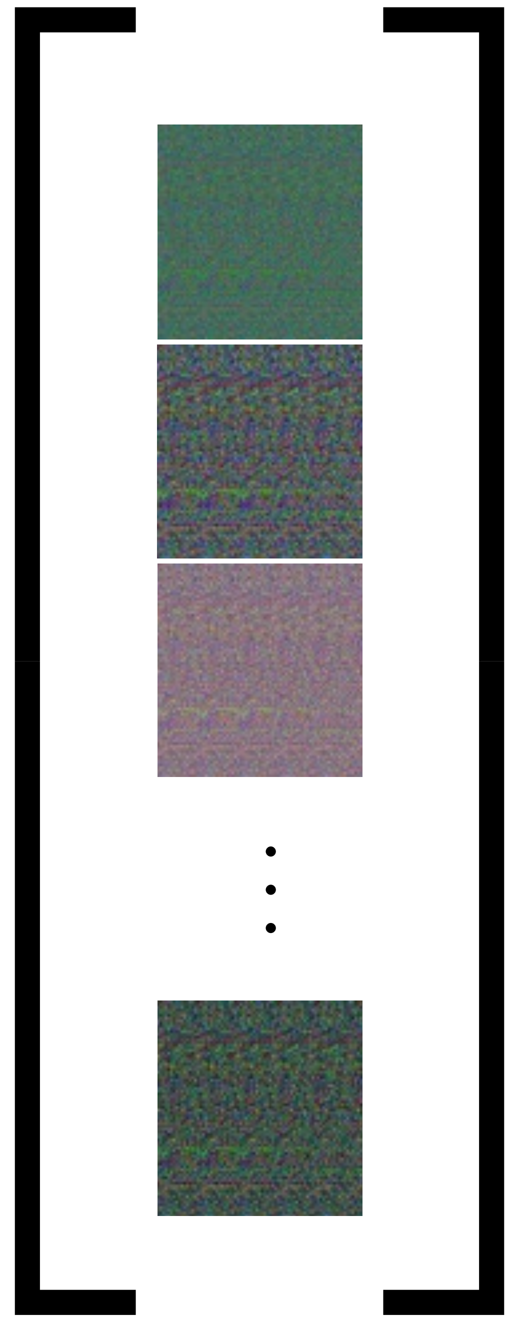
Sensing operator

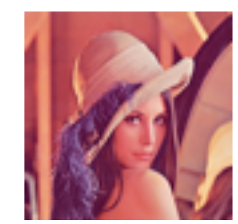
Meaningful information

N

Compressed sensing

$y =$





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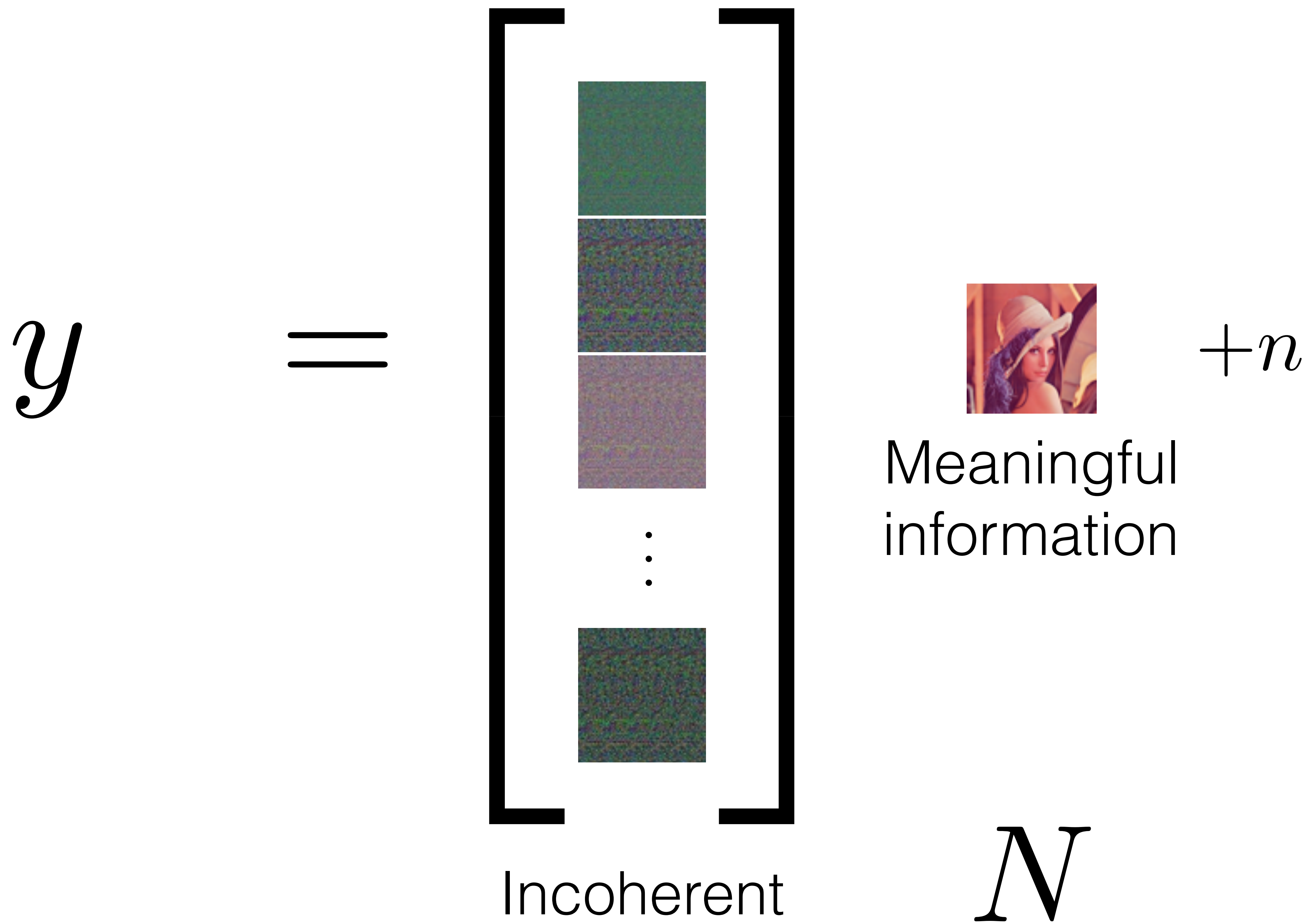
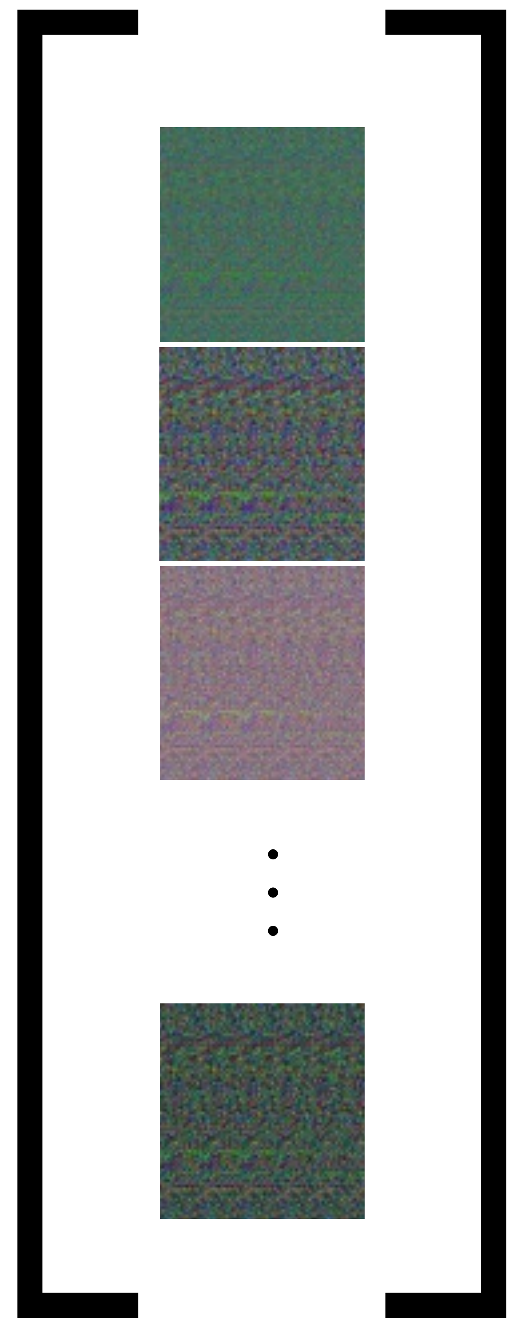
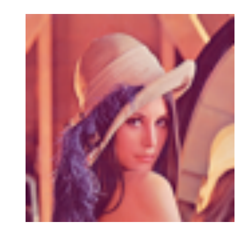


Compressed sensing

$y =$





$+n$

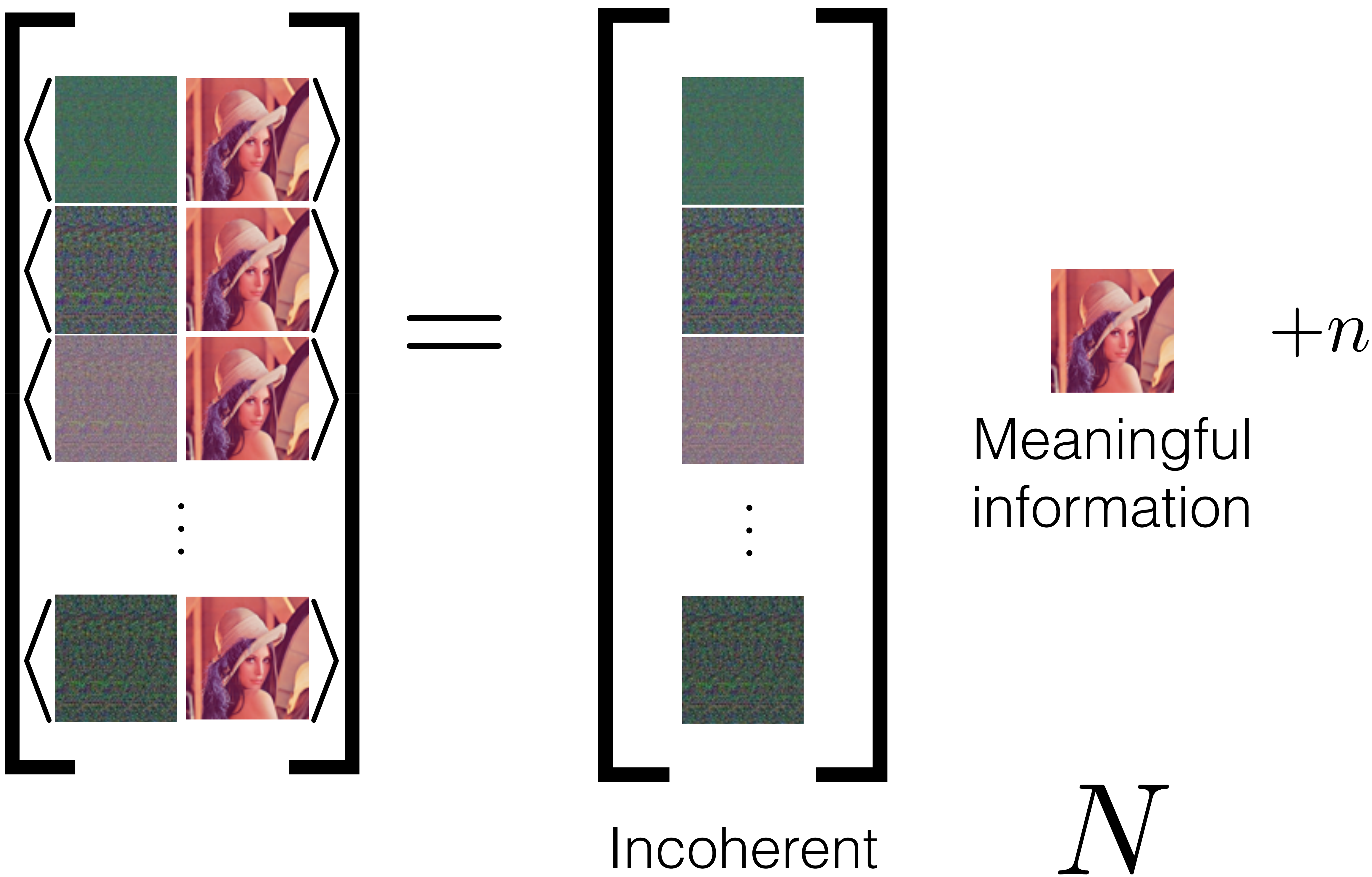
Compressive measurements

Incoherent

Meaningful information

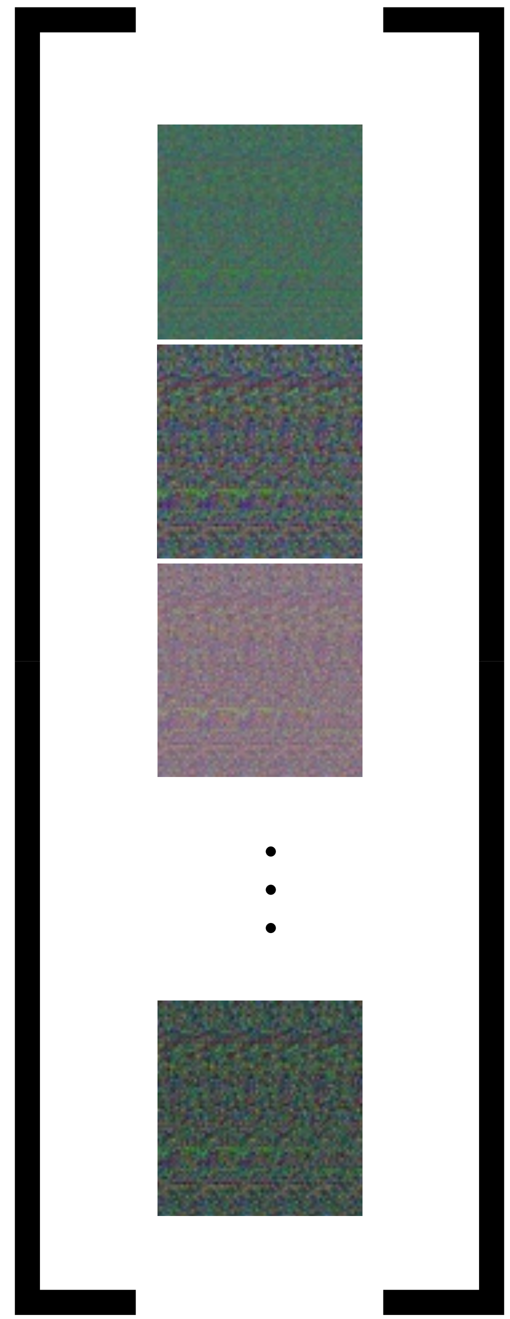
N

Compressed sensing



Compressed sensing

$$\begin{bmatrix} 1.212 \\ -5.235 \\ 3.623 \\ \vdots \\ -0.987 \end{bmatrix} =$$



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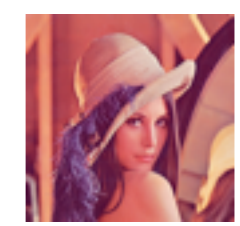


Meaningful information

$+n$

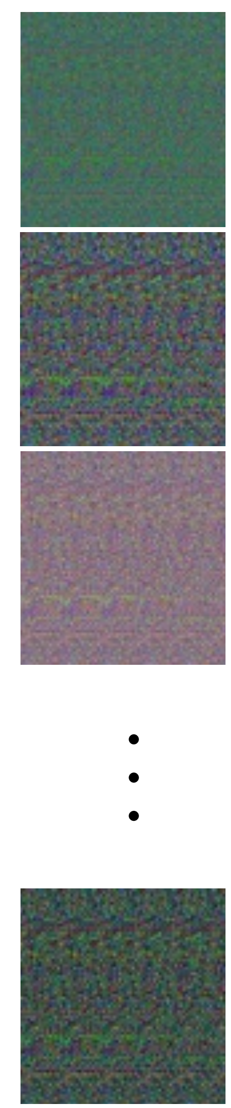
N

Compressed sensing

1.212
-5.235
3.623
⋮
-0.987

$M \ll N$

=



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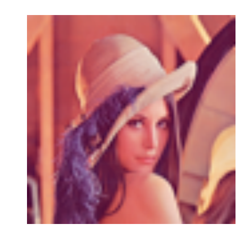

Meaningful information

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Synthesis reconstruction

$$x^\star = \arg\min_u \|u\|_0 \quad \text{s.t.} \quad \|\Phi u - y\| \leq \epsilon$$

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"The **sparsest** u that matches
the compressive measurements"

Synthesis reconstruction

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Hard !

"The **sparsest** u that matches the compressive measurements"

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$$x^\star = \arg\min_u \|u\|_1 \quad \text{s.t.} \quad \|\Phi u - y\| \leq \epsilon$$

Convex :-)

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not sparse

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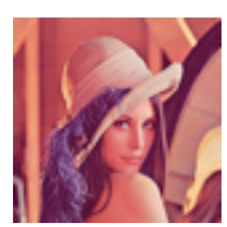
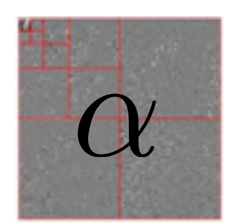
 $= \Psi$  α sparse :-)

Ψ orthonormal

Synthesis reconstruction

$$x^\star = \Psi \arg\min_{\alpha} \|\alpha\|_1 \quad \text{s.t.} \quad \|\Phi\Psi\alpha - y\| \leq \epsilon$$

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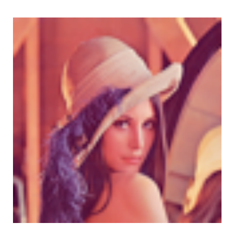
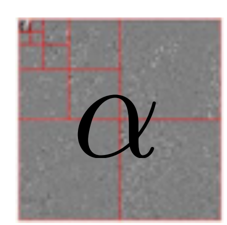


$= \Psi$



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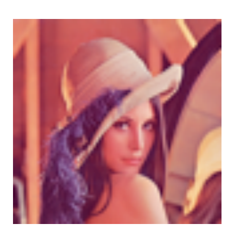
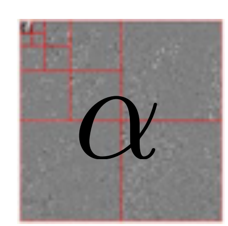
Ψ orthonormal

Synthesis reconstruction

$$x^\star = \Psi \arg\min_{\alpha} \|\alpha\|_1 \quad \text{s.t.} \quad \|\Phi\Psi\alpha - y\| \leq \epsilon$$

Convex :-)

“The **sparsest** α that matches the compressive measurements after some inverse transform Ψ”

 $= \Psi$  α sparse :-)

Ψ orthonormal

RIP and guaranties

Candès, Romberg, Tao (2006)

$\Theta = \Phi\Psi$ respects the RIP with constant δ_K if for all K-sparse α

$$\sqrt{1-\delta_K}\|\alpha\| \leq \|\Theta\alpha\| \leq \sqrt{1+\delta_K}\|\alpha\|$$

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Every subset of K or fewer columns is approximately orthonormal

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Sparse signals are not in the null space of Θ

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Small isometry constant δ if Φ is **incoherent** with the sparsity basis Ψ

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Holds for **randomised** sensing:
Gaussian, Bernoulli, random Fourier ensembles,…

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Holds for **randomised** sensing:
Gaussian, Bernoulli, random Fourier ensembles,…

With high probability if $M = \mathcal{O}\left(K \log^p N\right) \quad \ll N$

small integer

RIP and guaranties

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$$\alpha^\star = \arg\min_{\alpha} \|\alpha\|_1 \quad \text{s.t.} \quad \|\Theta\alpha - y\| \leq \epsilon$$

Θ is RIP with $\delta_{2K} < 0.4652$

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$$\|\alpha - \alpha^\star\|_2 \leq C_0 \frac{\|\alpha - \alpha_K\|_1}{\sqrt{K}} + C_1\epsilon$$

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best K-terms approximation

$$\|\alpha - \alpha^\star\|_2 \leq C_0 \frac{\|\alpha - \alpha_K\|_1}{\sqrt{K}} + C_1 \epsilon$$

depend on δ_{2K}

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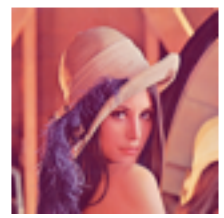
best K-terms approximation in Ψ

$$\|x - x^\star\|_2 \leq C_0 \frac{\|x - x_K\|_1}{\sqrt{K}} + C_1 \epsilon$$

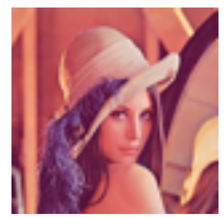
depend on δ_{2K}

Motivation

 $= \Psi$  α

orthonormal

Motivation

 $= \Psi$  α

orthonormal

What if Ψ is an *overcomplete* dictionary?

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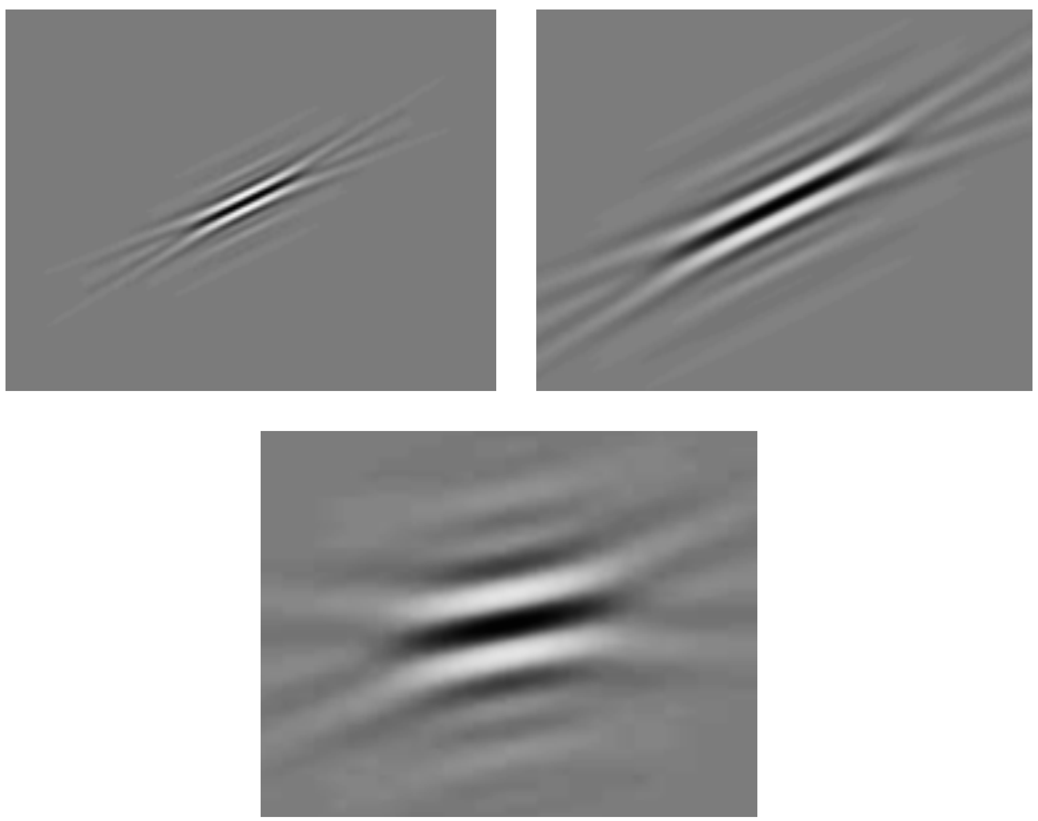
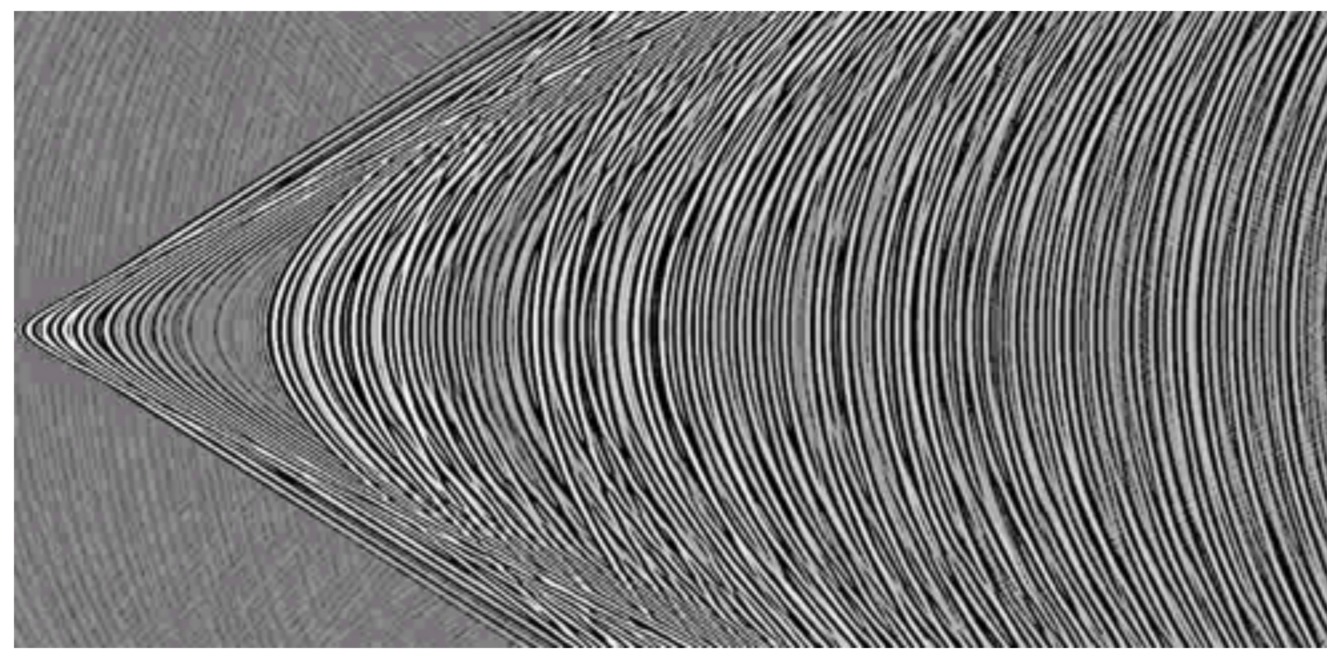




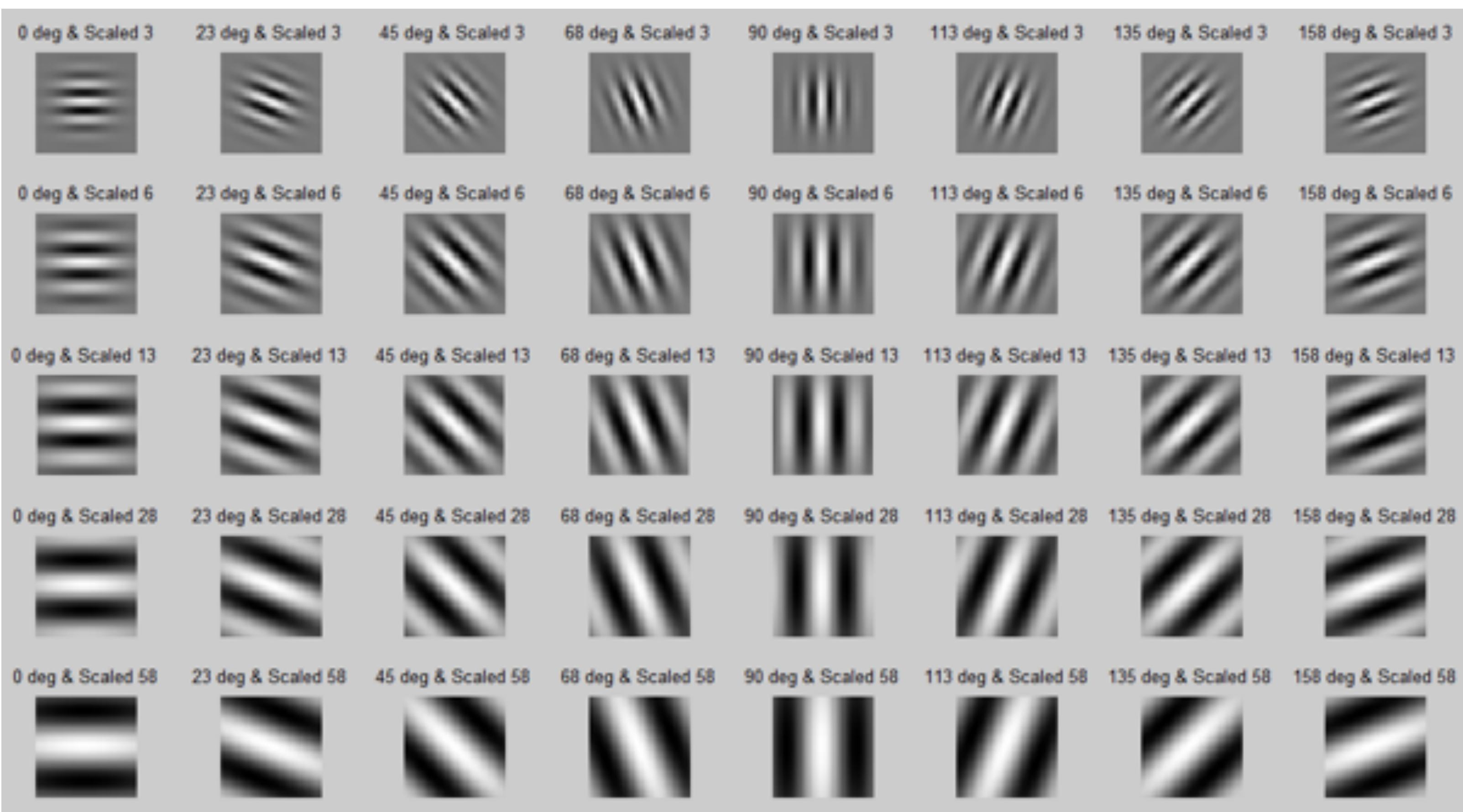
Fast Discrete Curvelet Transforms, Candès, Demanet, Donoho, Ying

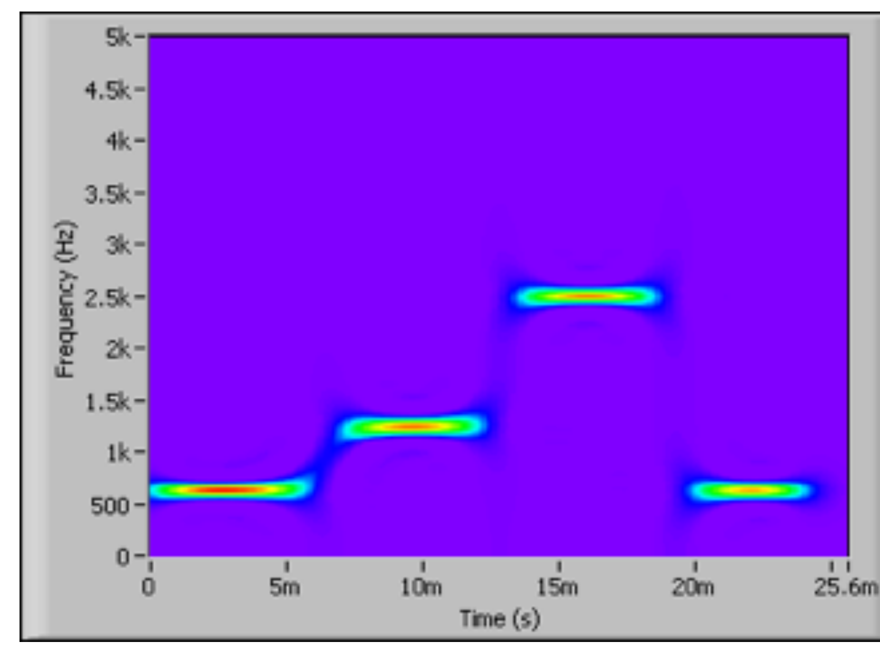
curvelets (for objects with edges)

Motivation

What if Ψ is an *overcomplete* dictionary?

Some sparsifying transforms do not have any orthobasis





time-frequency atoms (Gabor representation)

Motivation

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➜ we must work with tight frames

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frame

$$A\|v\|_2^2 \leq \|\Psi v\|_2^2 \leq B\|v\|_2^2$$

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➔ we must work with tight frames

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$$A\|v\|_2^2 \leq \|\Psi v\|_2^2 \leq B\|v\|_2^2$$

tight frame

$$\|\Psi v\|_2^2 = \tau\|v\|_2^2 \quad (A = B = \tau)$$

Motivation

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$$A\|v\|_2^2 \leq \|\Psi v\|_2^2 \leq B\|v\|_2^2$$

tight frame

$$\|\Psi v\|_2^2 = \tau \|v\|_2^2 \quad (A = B = \tau)$$

$$< v, \Psi^T \Psi v >= \tau < v, v >$$

$$\Psi^T \Psi = \tau$$

Motivation

What if Ψ is an *overcomplete* dictionary?

Overcomplete representation are flexible and convenient.

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Help reducing artifacts and MSE in

Deconvolution

Tomography

Signal denoising

…

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Entries of Ψ (atoms) are correlated

$\Theta = \Phi\Psi$ may not be RIP anymore

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Design Φ according to Ψ ➔ Loss of universality

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What if Ψ is an *overcomplete* dictionary?

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~~Design Φ according to Ψ → Loss of universality~~

see the work of P. Randall

Build a **new theory** for coherent overcomplete dictionary

Other examples

Other examples

Oversampled DFT

frequencies may be over smaller intervals or intervals of varying length

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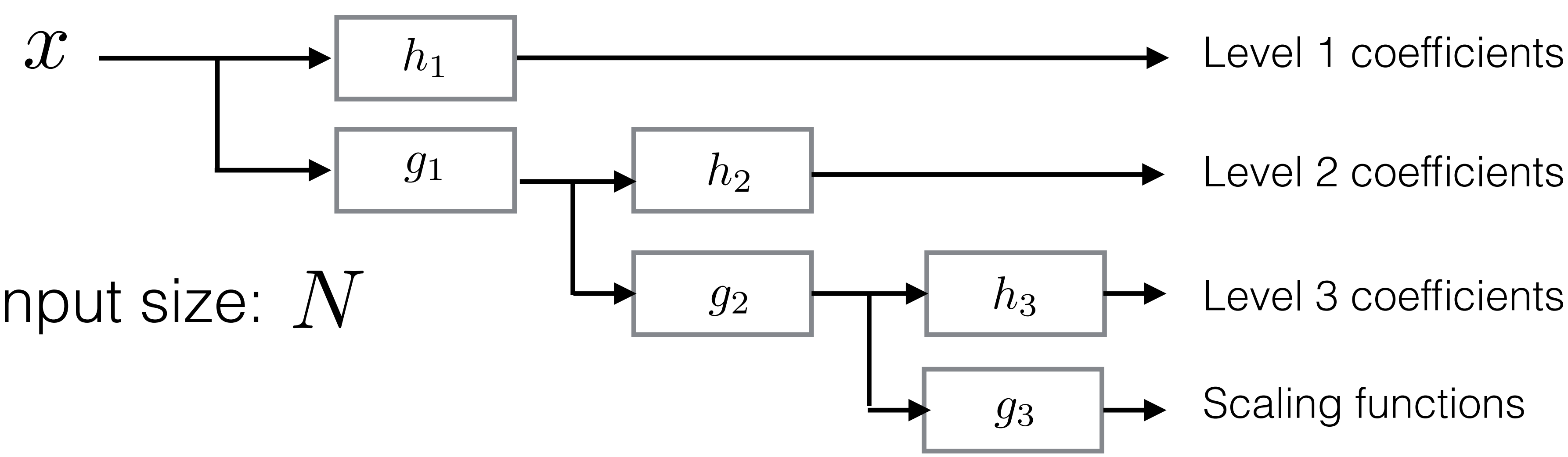
Oversampled DFT

frequencies may be over smaller intervals or intervals of varying length

Undecimated Wavelet transform



input size: N

Other examples

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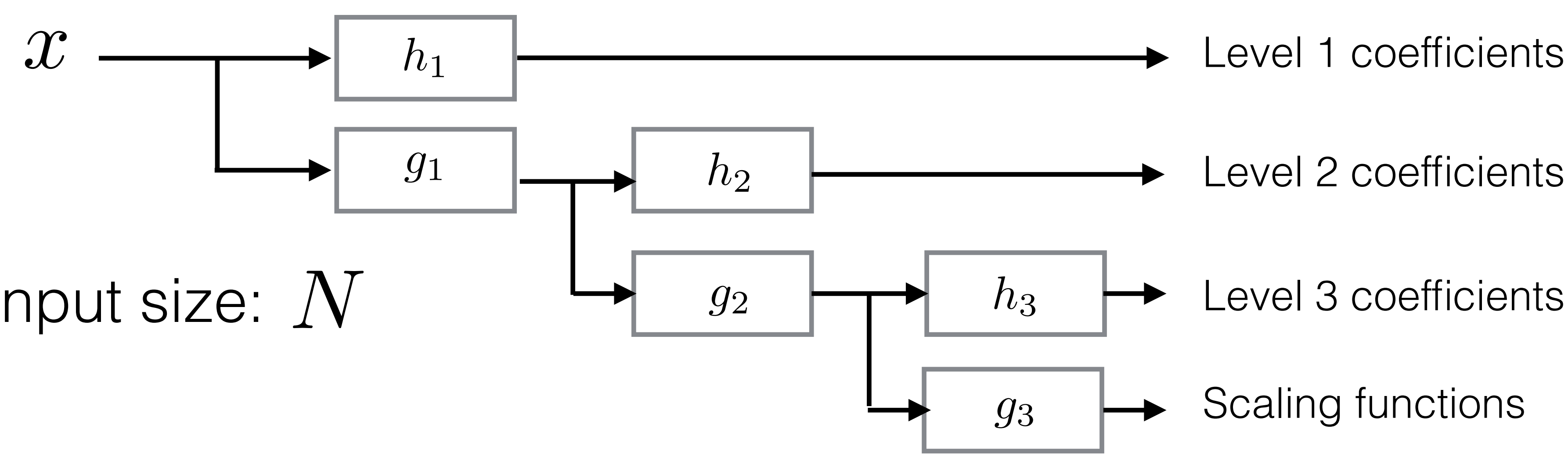

input size: N

no decimator

→ output size: $4N$

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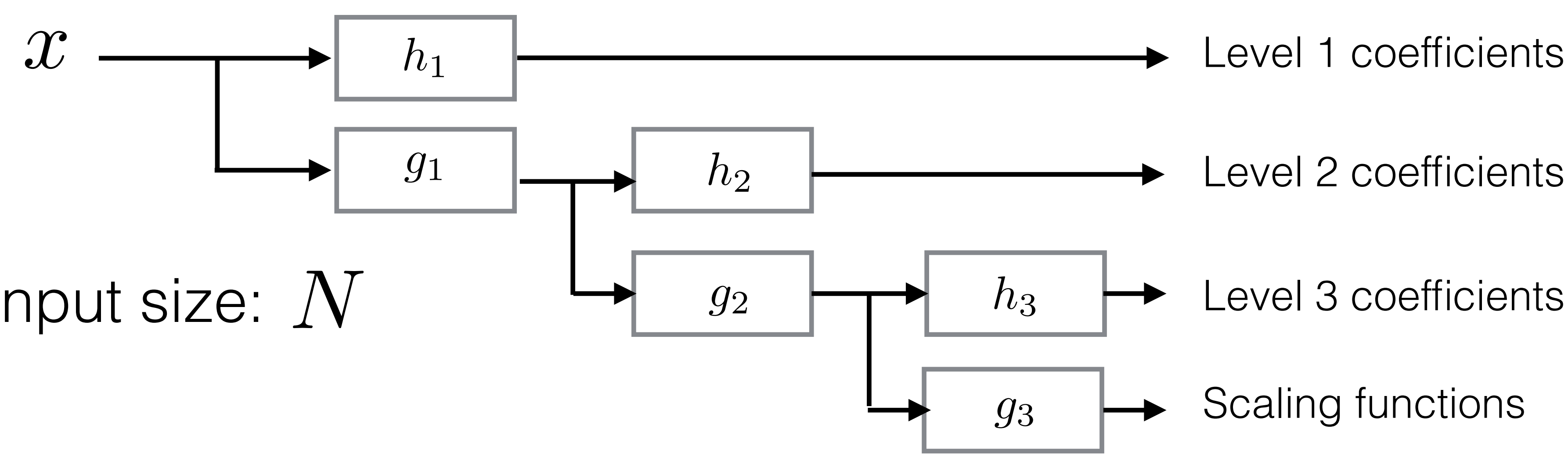

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translation invariant

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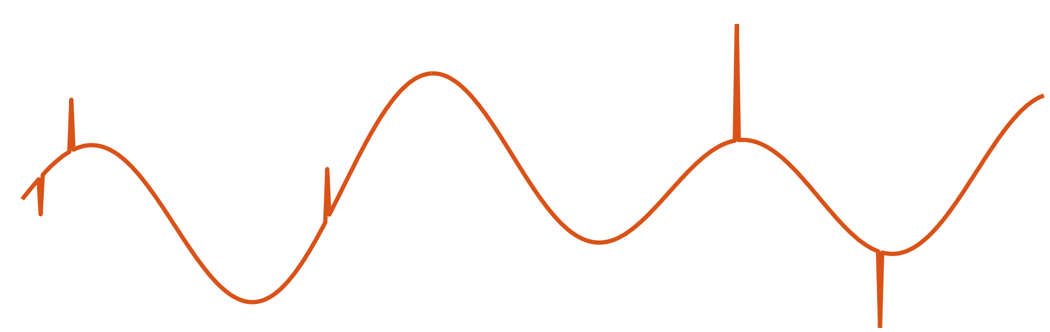
Concatenations

e.g., DFT with canonical basis



Is incoherence needed?

Coherence of a matrix

$$\mu\left(\Theta\right) = \max_{j<k} \frac{|\langle\theta_i, \theta_j\rangle|}{\|\theta_i\|_2\|\theta_j\|_2}$$

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Maximal off-diagonal entry of the normalised Gram matrix

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Maximal correlation between two columns

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Simple example :

let $\quad \Phi = \mathrm{Id} \quad \alpha_1 = e_1$

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Do we care?

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Do we care?

The objective is **not** to find α but to find x

→ maybe not!

Analysis reconstruction

Recall the synthesis reconstruction

$$x^\star = \Psi \arg\min_\alpha \|\alpha\|_1 \quad \text{s.t.} \quad \|\Phi\Psi\alpha - y\| \leq \epsilon$$

We search the coefficient vector that synthesizes x

Analysis reconstruction

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Instead we propose

$$x^\star = \arg\min_x \|\Psi^T x\|_1 \quad \text{s.t.} \quad \|\Phi x - y\| \leq \epsilon$$

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Special case : Φ Gaussian with $M = \mathcal{O}\left(K \log N\right)$

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Ψ is an arbitrary tight frame

Then

$$\|x - x^\star\|_2 \leq C_0 \frac{\|\Psi^T x - (\Psi^T x)_K\|_1}{\sqrt{K}} + C_1 \epsilon$$

Holds, even if Ψ is **maximally coherent** !

Implications

$$\|x - x^\star\|_2 \leq C_0 \frac{\|\Psi^T x - (\Psi^T x)_K\|_1}{\sqrt{K}} + C_1 \epsilon$$

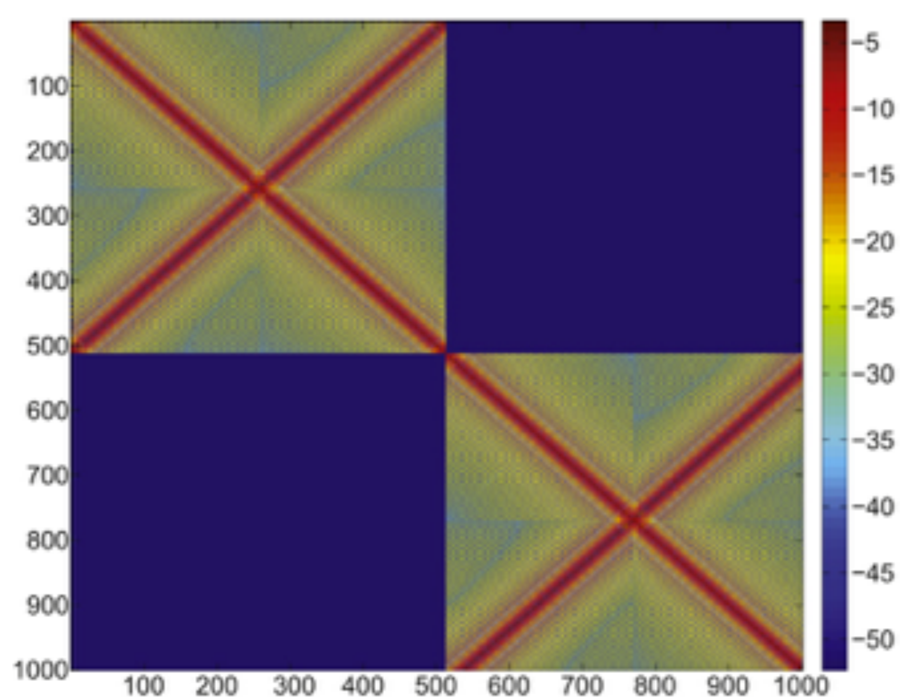
tail of the signal

small if $\Psi^T \Psi$ is "reasonably sparse" and

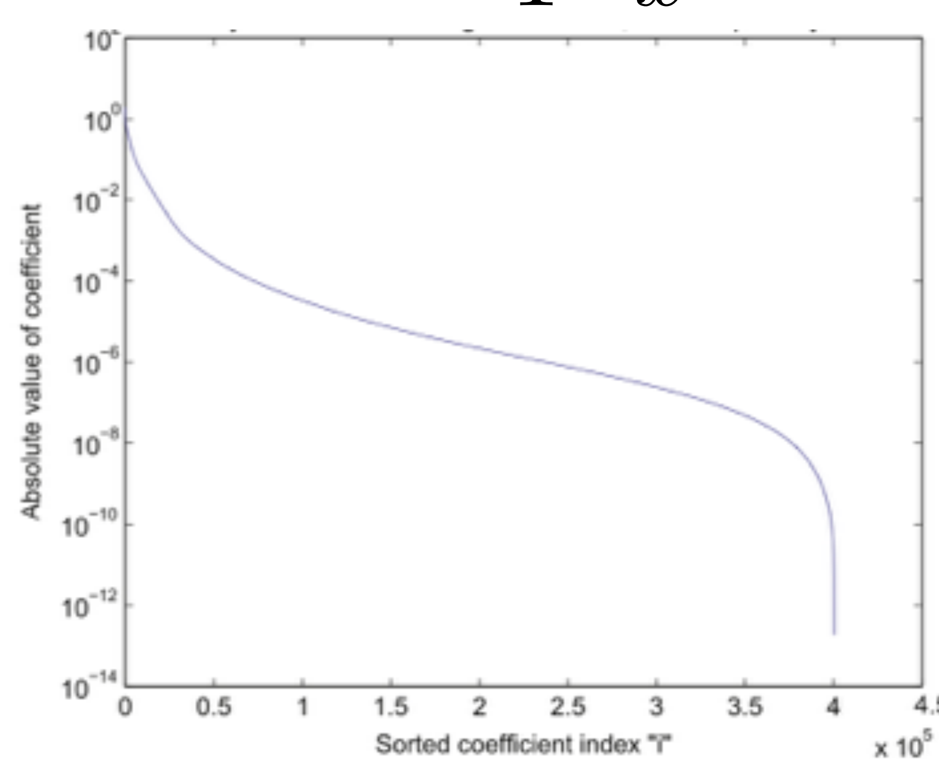
$\exists \alpha$ nearly sparse such that $x = \Psi \alpha$

Example :
Gabor dictionary with Gaussian windows
(see numerical results)

$\Psi^T \Psi$



$\Psi^T x$

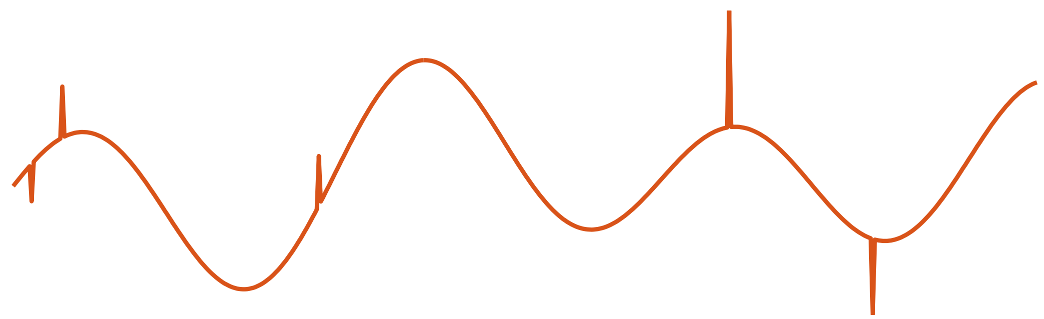


Implications

Works for a lot of dictionaries

Ovesampled DFT, Gabor frames, UDWT,
Curvelet frames,….

Not for concatenations of two orthobases



Neither sparse in DFT analysis nor in Canonical basis

General case

D-RIP adapted to the dictionary Ψ (with constant δ_K)

$\Sigma_K :=$ the union of subspaces spanned by all subsets of K columns of Ψ , i.e., the image under Ψ of all K-sparse vectors

$\forall x \in \Sigma_K$

$$\sqrt{1-\delta_K}\|x\| \leq \|\Phi x\| \leq \sqrt{1+\delta_K}\|x\|$$

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Any matrix satisfying the RIP will satisfy the D-RIP after randomizing the column signs

see Krahmer and Ward

Main result

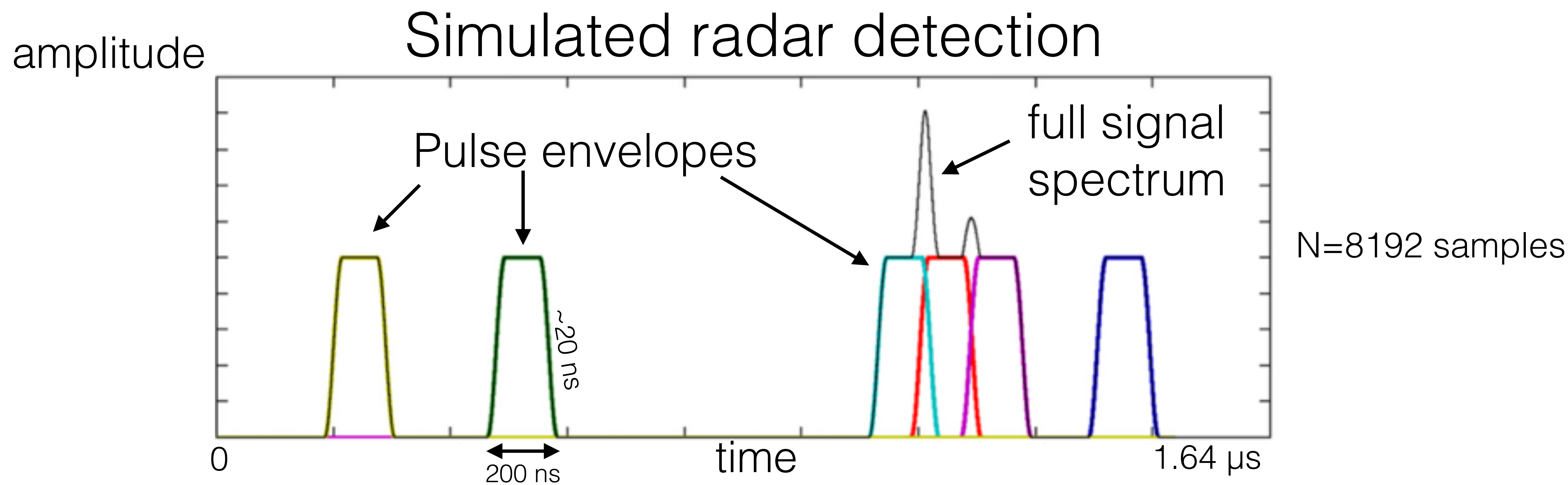
If Φ satisfies the D-RIP adapted to the dictionary Ψ with constant $\delta_{2K} < 0.08$ (or $\delta_{7K} \leq 0.6$)

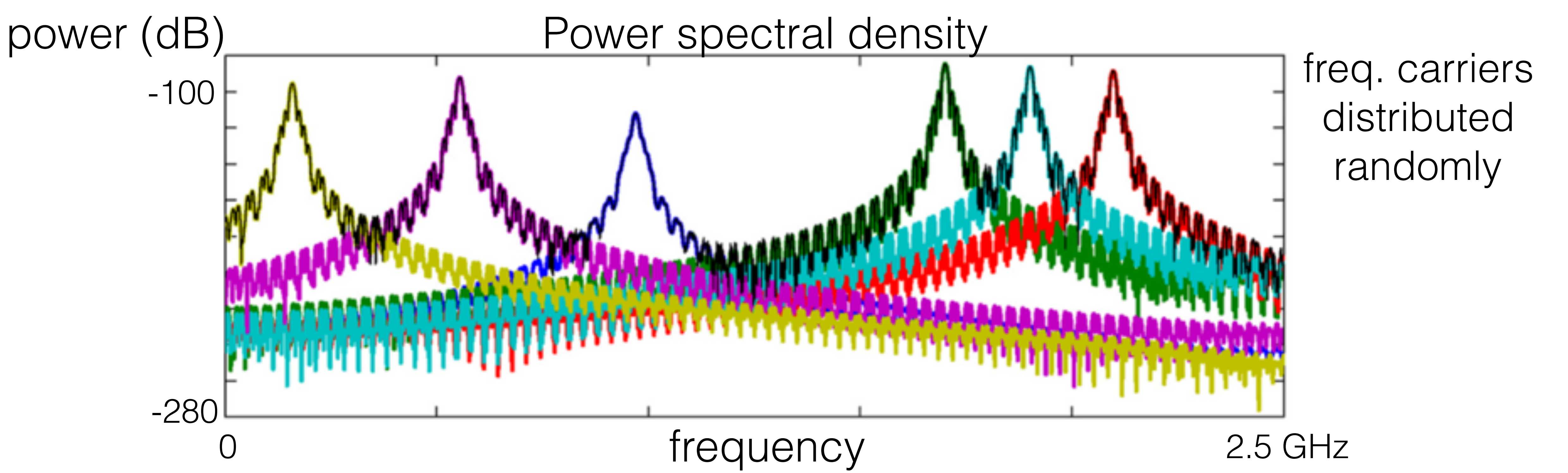
Then the solution of the analysis reconstruction satisfies

$$\|x - x^\star\|_2 \leq C_0 \frac{\|\Psi^T x - (\Psi^T x)_K\|_1}{\sqrt{K}} + C_1 \epsilon$$

depend on δ_{2K}

Numerical results





Numerical results

$N = 8192$

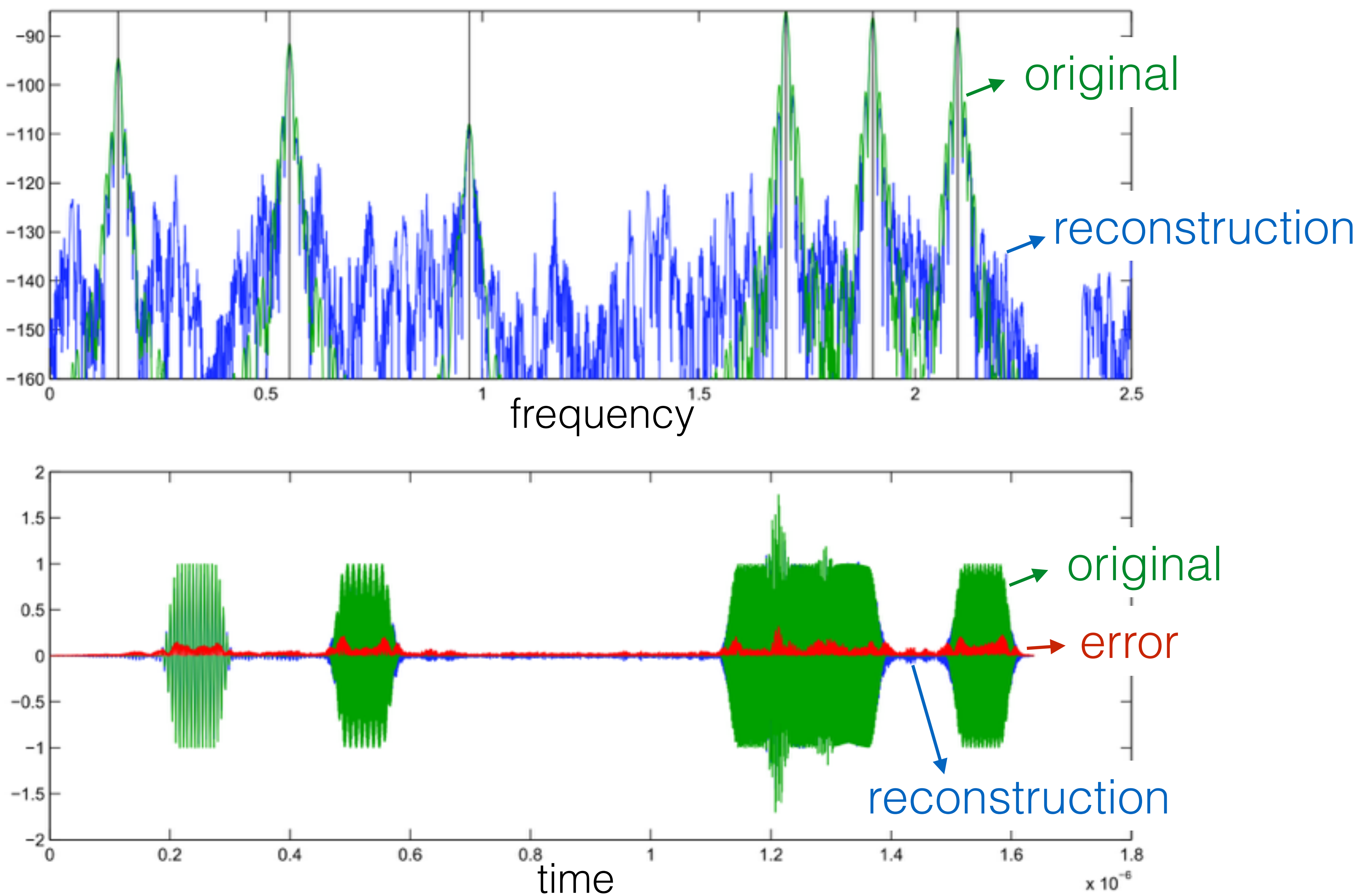
Φ $M \times N$ Gaussian $M = 400$

Ψ $N \times D$ Gabor dictionary $D \approx 60N$
Gaussian windows

x is not exactly sparse in Ψ because

- the pulse envelopes are not Gaussian;
- frequencies and arrival times sample from a continuous grid.

Analysis ℓ_1 reconstruction



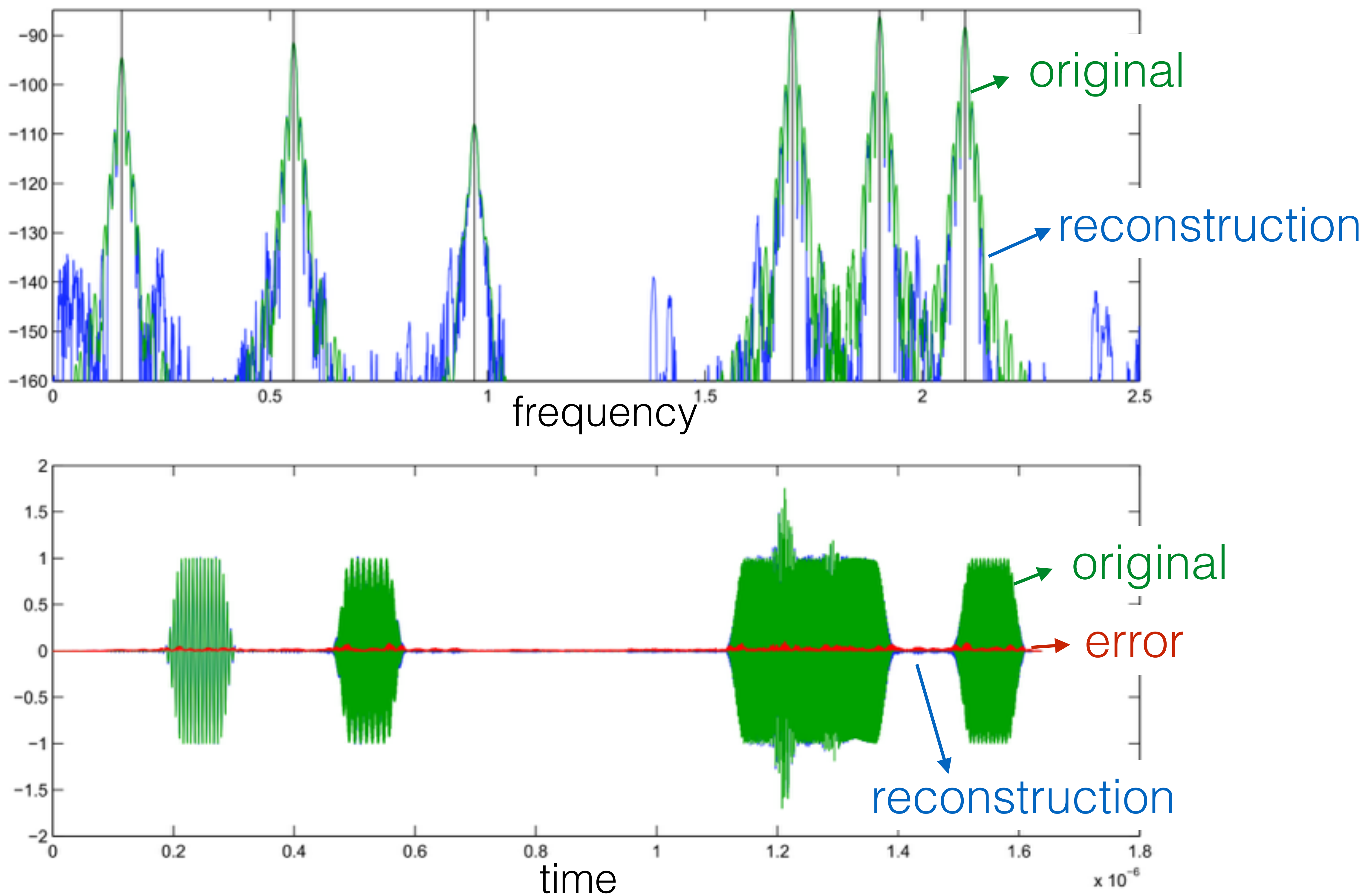
Reweighted ℓ_1 analysis

Enhanced method based on the original

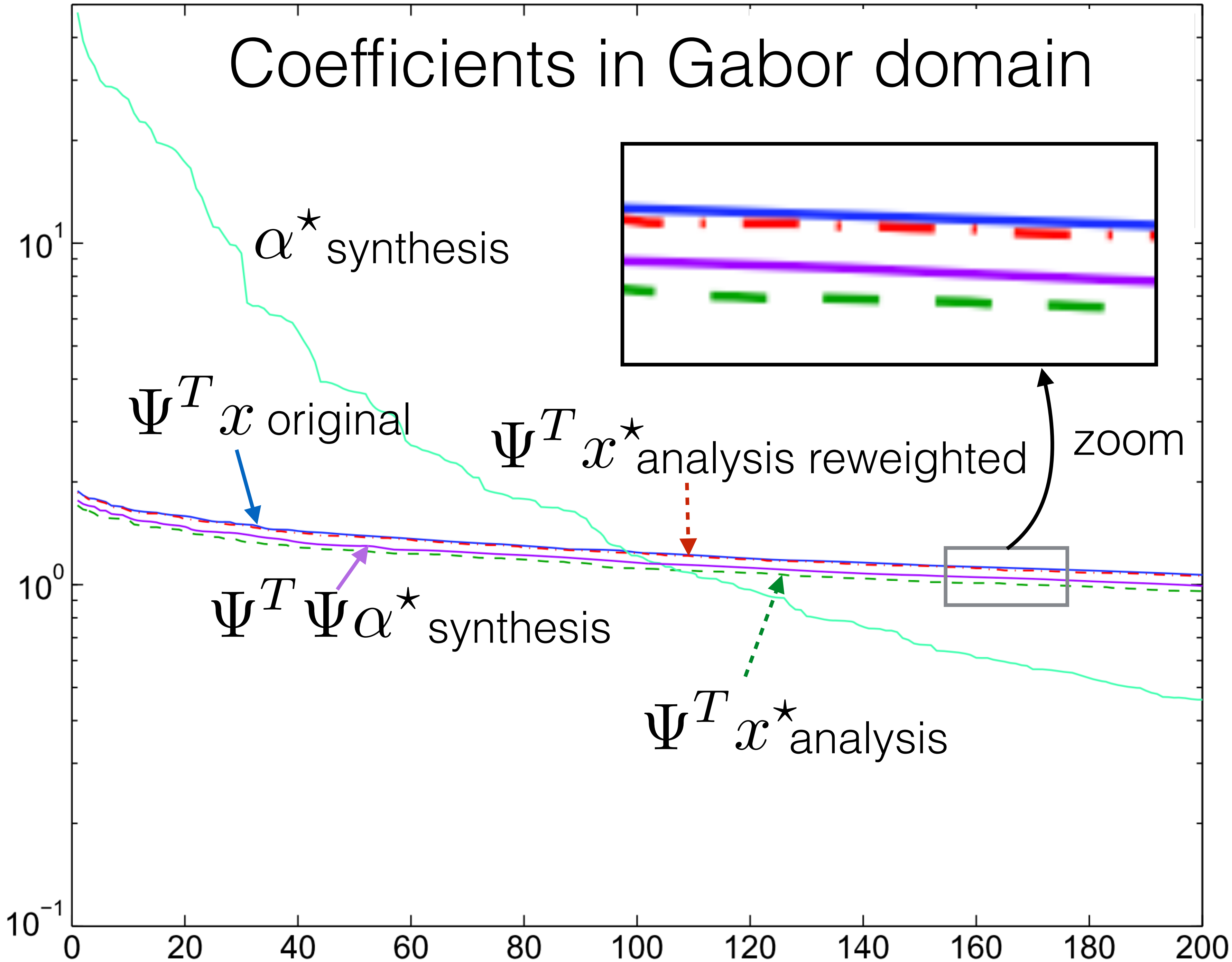
"solves **several** sequential weighted ℓ_1-minimization problems, each using weights computed from the solution of the previous problem"

Known to "outperform standard ℓ_1-minimization in many situations"

Reweighted ℓ_1 analysis

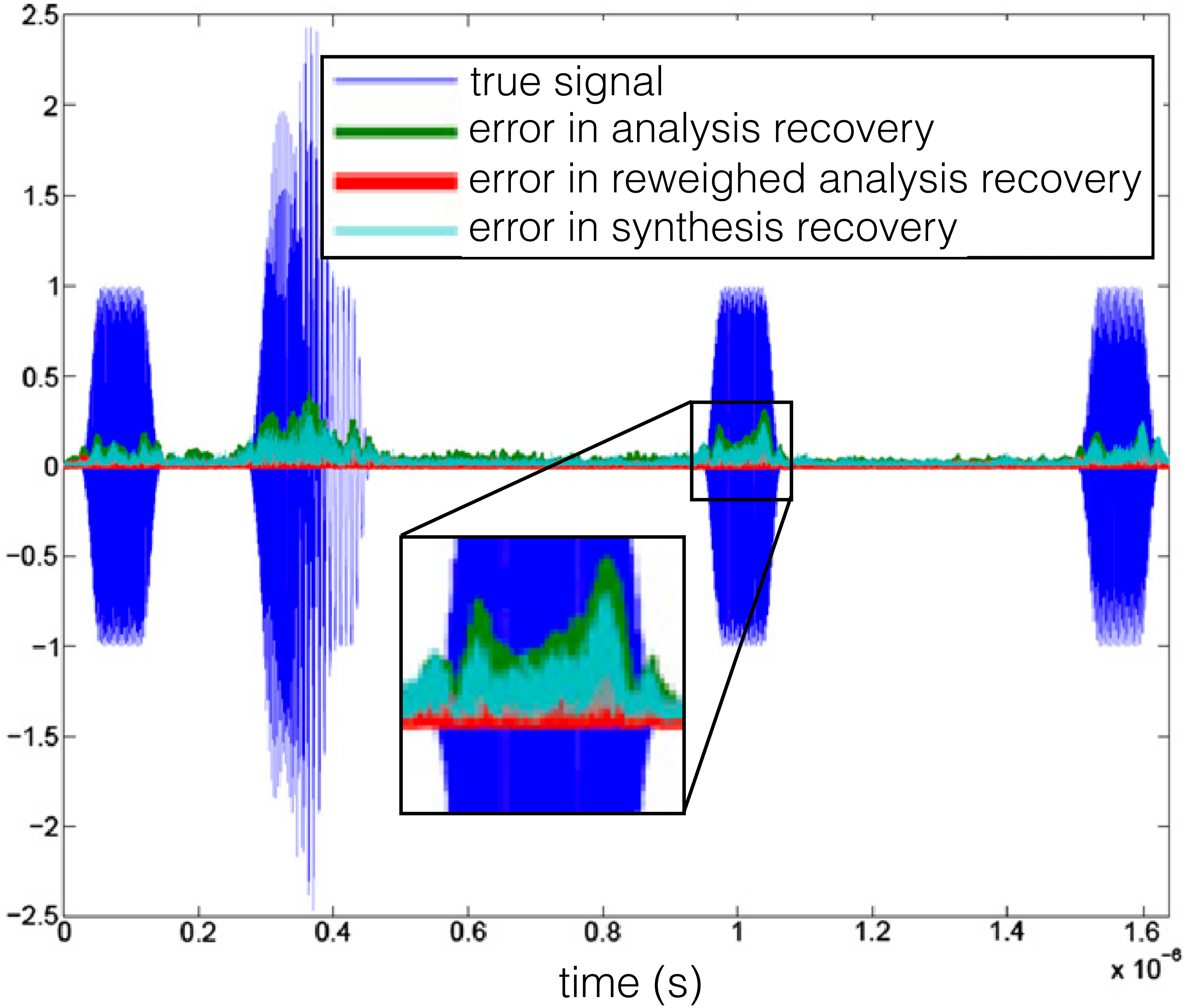


Comparison



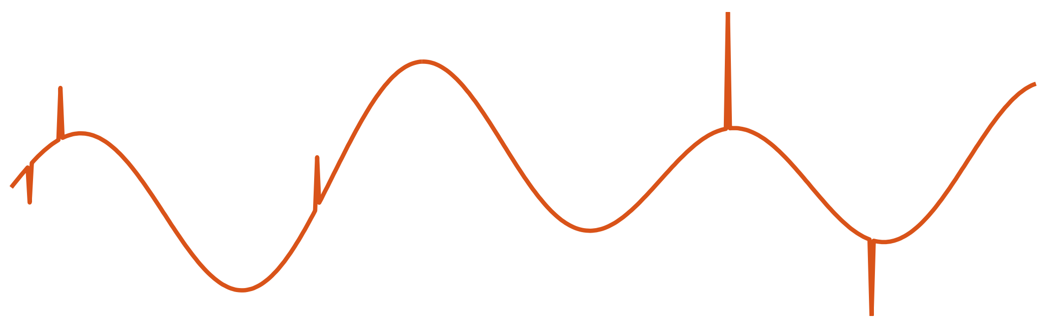
Coefficients in Gabor domain
$\alpha^\star$ synthesis
$\Psi^T x$ original
$\Psi^T x^\star$ analysis reweighted
zoom
$\Psi^T \Psi \alpha^\star$ synthesis
$\Psi^T x^\star$ analysis
10^1
10^0
10^{-1}
0
20
40
60
80
100
120
140
160
180
200

Comparison


true signal
error in analysis recovery
error in reweighed analysis recovery
error in synthesis recovery
2.5
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2.5
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
x 10^-6
time (s)

Discussions

How to deal with concatenations of orthobases?



$$x = x_1 + x_2$$

Split-analysis

$$(x_1^\star, x_2^\star) = \arg\min_{x_1, x_2} \|\Psi_1^T x_1\|_1 + \|\Psi_2^T x_2\|_1 \quad \text{s.t.} \quad \|\Phi(x_1 + x_2) - y\| \leq \epsilon$$

Synthesis may also work in this case

Very different geometrical properties.
Performs on different classes of signal than analysis.

2 comments from Nam *et al.*

S. Nam, M. E. Davies, M. Elad, and R. Gribonval, "The cosparse analysis model and algorithms," Appl. Comput. Harmon. Anal., vol. 34, no. 1, pp. 30–56, Jan. 2013.

As soon as $\Theta = \Phi\Psi$ respects the standard RIP, the analysis recovery tends to perform in general worse than the synthesis.

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Solution of the synthesis problem (noiseless case)

$$\|\Psi\alpha^\star - x\| \leq C_s \frac{\|\Delta_0(x) - (\Delta_0(x))_K\|_1}{\sqrt{K}}$$

$\Delta_0(x)$ sparsest representation of x in Ψ in particular sparsest than $\Psi^T x$

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synthesis error term — analysis error term

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Cosparsity : number of zeros in the analysis domain $\Psi^T x$

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but $\delta_{7(0.15N+1)} = \delta_{1.05N+7} \geq 1$ (because $\Sigma_N = \mathbb{R}^N$)

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The requirement of the theorem cannot be met !

Take home messages

- Redundant dictionaries are useful in compressed sensing too.
- Random sensing matrices still work (D-RIP).
- Analysis formulation may help for a lot of problems.
- Synthesis formulation is more suited for a lot of other problems.
- The theoretical bound may be further optimized.
- The D-RIP has its limits and the cosparse analysis can also help for theoretical works.

Thank you !

Analysis prior with redundant dictionaries for Compressed Sensing

Kévin Degraux

After the article

E. J. Candès, Y. C. Eldar, D. Needell, and P. Randall,
"Compressed sensing with coherent and redundant dictionaries,"
Appl. Comput. Harmon. Anal., vol. 31, no. 1, pp. 59–73, Jul. 2011.