Continuous Parameter Estimation from Compressive Samples

Prasad Sudhakar

Postdoctoral researcher, ICTEAM school

Université catholique de Louvain, Belgium

09 October, 2013

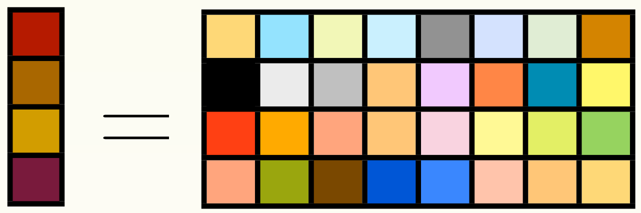
ISP Seminars, UCL

Avec le soutien de la DGO6
Département du Développement Technologique

Compressed sensing

- Linear measurements of $\boldsymbol{s} \in \mathbb{C}^N$

$\boldsymbol{\Phi} \in \mathbb{C}^{M \times N}$

$\boldsymbol{y} \in \mathbb{C}^M$  $\boldsymbol{s} + \boldsymbol{n}$

$M < N$

Under-determined system

- K-sparse signal

$$\underset{\in \mathbb{C}^N}{\boldsymbol{s}} = \boldsymbol{\Psi} \underset{\in \mathbb{C}^N}{\boldsymbol{\alpha}}$$

$\#\{j : \alpha_j \neq 0\}$ $\|\boldsymbol{\alpha}\|_0 \leq K \ll N$

- To obtain $\boldsymbol{s}$, compressive sensing advocates solving

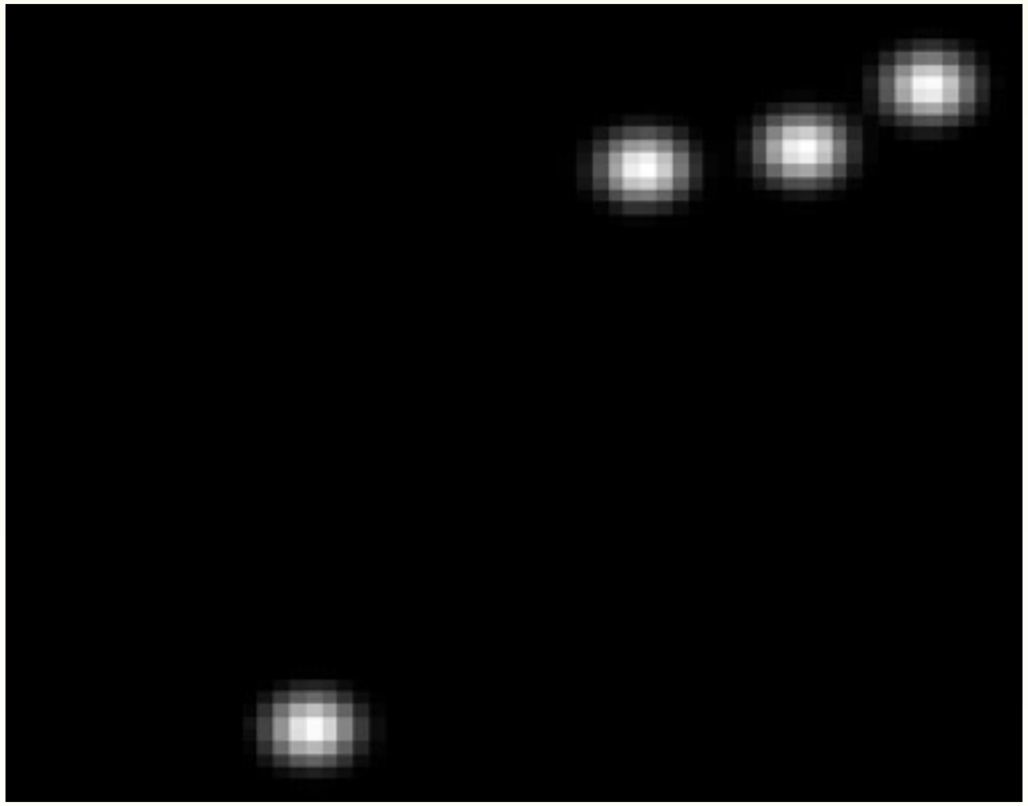
$$\widehat{\boldsymbol{\alpha}} := \underset{\widetilde{\boldsymbol{\alpha}} \in \mathbb{C}^N}{\arg\min} \|\widetilde{\boldsymbol{\alpha}}\|_1 \text{ subject to } \|\boldsymbol{y} - \boldsymbol{\Phi}\boldsymbol{\Psi}\widetilde{\boldsymbol{\alpha}}\|_2 \leq \epsilon. \qquad \textbf{(P1)}$$

- Succeeds, if M is ‘large’ enough, and $\boldsymbol{\Phi}\boldsymbol{\Psi}$ is nice

Useful information without reconstruction

- Sometimes, it is sufficient if we can extract specific information about $\boldsymbol{s}$

- Example - Localisation of *known* patterns in an image



$\boldsymbol{s}$

- *Biological applications*
- *1D: speech/audio processing*

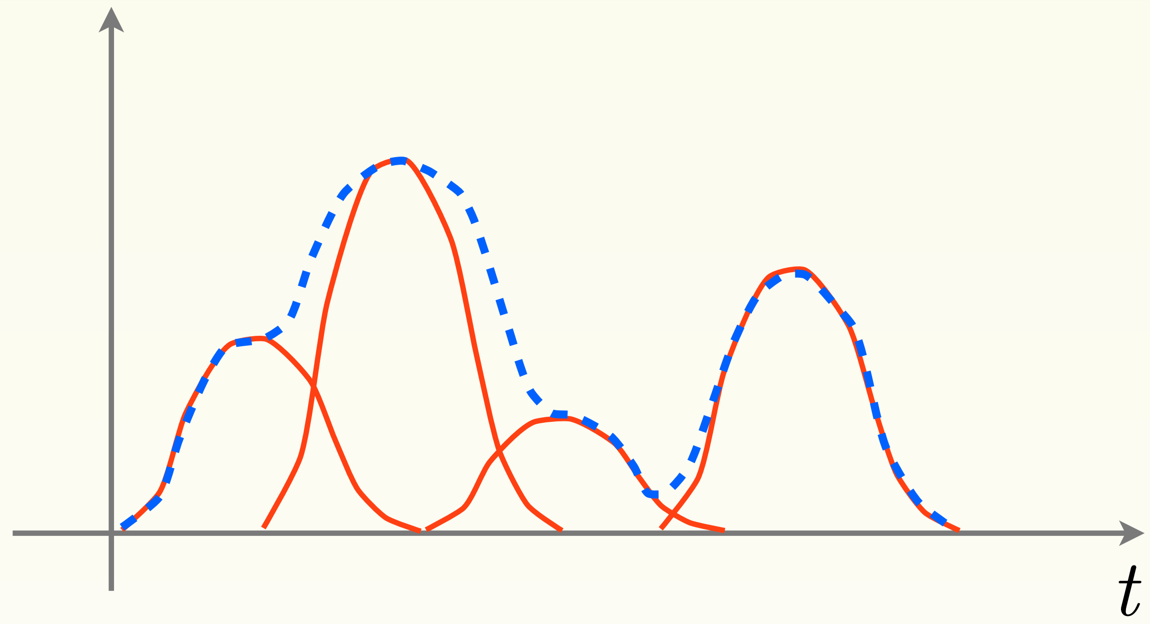
- Location is a continuous parameter
- Only compressive samples are available

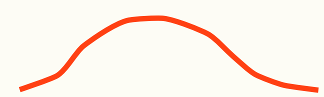
Signal model

- Continuous signal model

$$s_c(t) = \sum_{j=1}^{K} \alpha_j f_c(t - \tau_j)$$

- $\{\alpha_j\}, \{\tau_j\}$: unknown amplitude and delay





- $f_c(t)$: known signal

- Fixed sampling grid

- $\boldsymbol{f}_{\tau_j} \in \mathbb{R}^N$: vector of samples of $f_c(t - \tau_j)$
- $\boldsymbol{s} = \sum_{j=1}^{K} \alpha_j \boldsymbol{f}_{\tau_j}$: discrete signal
- $\{\alpha_j\}, \{\tau_j\}$: continuous

Discrete finite signal dictionary

- Suppose $K \ll N$,

  $\boldsymbol{s}$ is sparse w.r.t a dictionary containing by ALL admissible translations of $f_c(t)$

  But, impossible!

- So, settle for a finite equivalent

  given a finite sampling of delay parameters $\{\delta_i\}_{i=1}^{D}$, with step size Δ, build

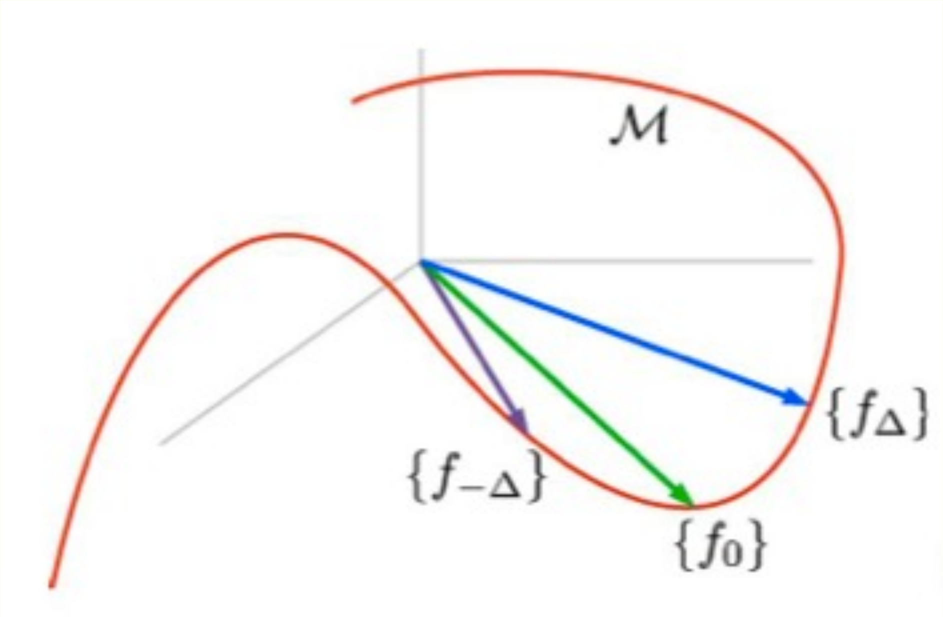
$$\boldsymbol{\Psi} = \left[\boldsymbol{f}_{\delta_1}, \boldsymbol{f}_{\delta_2} \dots \boldsymbol{f}_{\delta_D}\right]$$

  - Small $\Delta \longrightarrow$ strongly correlated columns

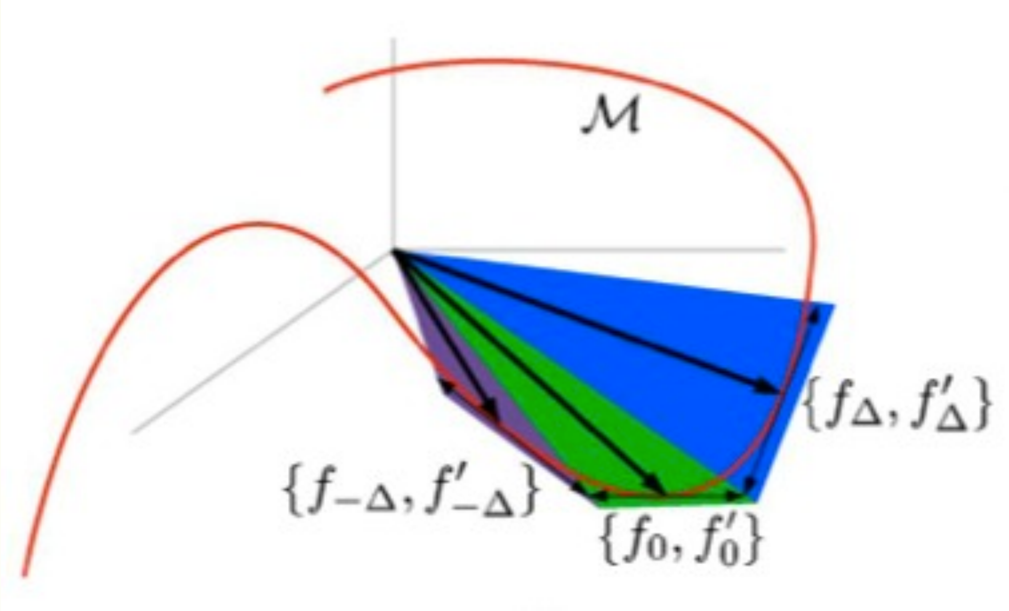
- Solving (**P1**) can only estimate delays with error in $[-\Delta/2, \Delta/2]$

From on grid to off grid delays

- Let $\mathcal{M}$: Set of all delayed and scaled signals
- Columns of $\mathbf{\Psi}$: linear subspace approximation of $\mathcal{M}$



Three consecutive columns of $\mathbf{\Psi}$



Taylor interpolation around each column

- Idea: Include interpolator functions in $\mathbf{\Psi}$

- Solve specialised $(\mathbf{P1})$ to obtain the off grid delays

Figures taken from the reference [Ekanadham 2011]

Polar interpolation 1/2

- If $f_c(t)$ is such that
  - $\|f_c(t-\delta_i)\|_2 = \|f_c(t-\delta_j)\|_2$ for all δ_i, δ_j : norm preservation
  - $\|f_c(t-\delta_i) - f_c(t-\delta_i+\delta)\|_2 = \|f_c(t-\delta_i) - f_c(t-\delta_i-\delta)\|_2$

    symmetric curvature

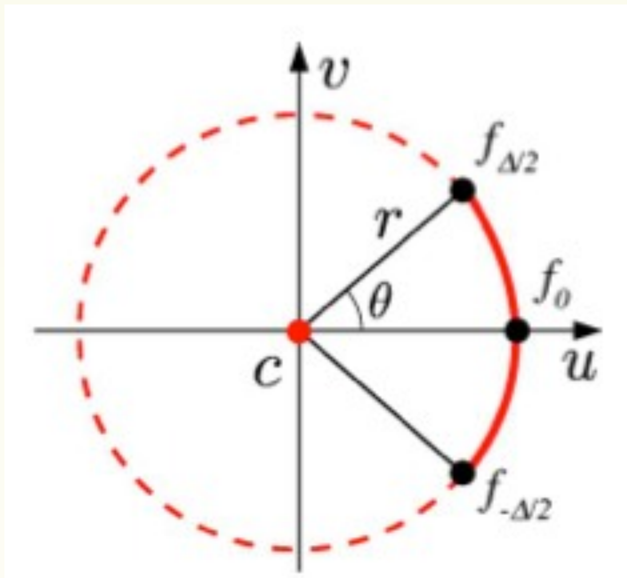
- Polar interpolation: Approximation by an arc of a circle
  - Any segment of $\mathcal{M}$ around δ_i can be written as

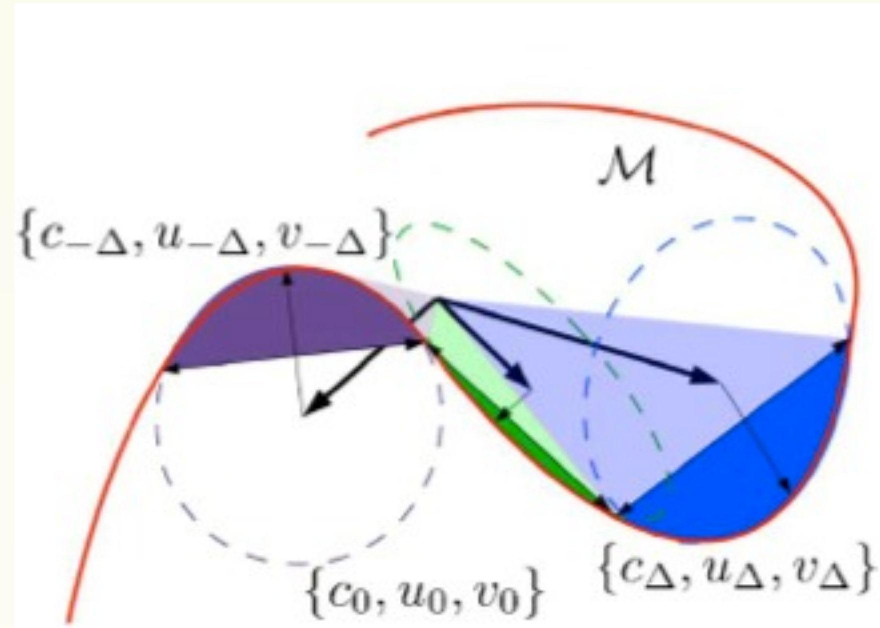
$$f_c(t-\tau) \approx c(t) + r\cos\left(\frac{2\tau}{\Delta}\theta\right)u(t) + r\sin\left(\frac{2\tau}{\Delta}\theta\right)v(t) \quad \tau \in [-\Delta/2, \Delta/2]$$

$$\begin{pmatrix} f_c(t-\delta_i+\Delta/2) \\ f_c(t-\delta_i) \\ f_c(t-\delta_i-\Delta/2) \end{pmatrix} = \begin{pmatrix} 1 & r\cos(\theta) & -r\sin(\theta) \\ 1 & r & 0 \\ 1 & r\cos(\theta) & r\sin(\theta) \end{pmatrix} \begin{pmatrix} c(t) \\ u(t) \\ v(t) \end{pmatrix}$$

$$r = \|f_c(t)\|_2 \quad \theta = \text{angle}(f_c(t-\delta_i), f_c(t-\delta_i+\Delta/2))$$

Polar interpolation 2/2





- Incorporate all the interpolating functions in the dictionary $\mathbf{\Psi}$
- Solve specialised (**P1**) to obtain the off grid delays

- Perform reconstruction oblivious of delay-scale model
- Estimate delays in signal domain (matched filtering, etc.)

Figures taken from the reference [Ekanadham 2011]

From proxy delays to actual delays

- Idea: Greedy approach
  - $\boldsymbol{y}_{res} = \boldsymbol{y}$
  - Select a column of $\boldsymbol{\Psi}$ that maximally correlates with residue
  - Use the obtained *proxy* delay δ_p and perform polar interpolation to obtain $\hat{\tau}_j$

$$\boldsymbol{y}_{res} \approx \boldsymbol{\Phi} \left[\boldsymbol{f}_{\delta_p - \Delta/2} \quad \boldsymbol{f}_{\delta_p} \quad \boldsymbol{f}_{\delta_p + \Delta/2}\right] \left(\begin{bmatrix} 1 & r\cos(\theta) & -r\sin(\theta) \\ 1 & r & 0 \\ 1 & r\cos(\theta) & r\sin(\theta) \end{bmatrix}^{-1} \right)^T \boldsymbol{x}$$

$$\boldsymbol{x} = \begin{bmatrix} \alpha_j \\ \alpha_j r \cos\left(\frac{2\tau\theta}{\Delta}\right) \\ \alpha_j r \sin\left(\frac{2\tau\theta}{\Delta}\right) \end{bmatrix}$$

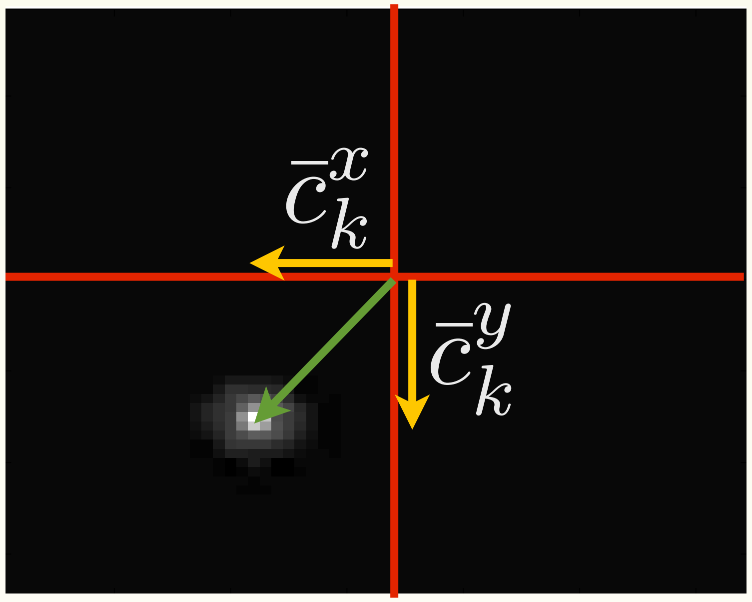
solved as least squares

$$\hat{\tau}_j = \delta_j + \arctan\left(\frac{\hat{x}_3}{\hat{x}_2}\right) \frac{\Delta}{2\theta}$$

- Obtain new residue incorporating the interpolated function
- Repeat K times

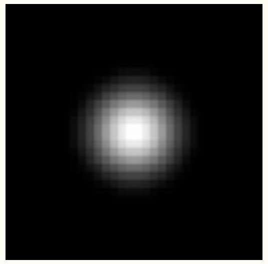
Example: Compressive deflectometry 1/2

- Several images $\{\boldsymbol{s}_k\}$



Location of dominant deflection.

- Each $\boldsymbol{s}_k$ characterised by $\widehat{\tau}_k = (\bar{c}_k^x, \bar{c}_k^y)^T$.

- $\boldsymbol{f}_\tau^\rho$: 2D Gaussian 

with radius ρ, ***translated*** by τ.

- To obtain proxy translation parameter

    - Matched filtering $\widehat{\tau}_k = \arg\max_{\tau} |\langle \boldsymbol{s}_k, \boldsymbol{f}_\tau^\rho \rangle|$.

- Efficiently implemented as convolution

Example: Compressive deflectometry 2/2

- Compressive matched filtering ***(Smashed filtering)***

$$\boldsymbol{y}_k \in \mathbb{R}^M, \widetilde{\tau}_k = \arg\max_{\tau} |\langle \boldsymbol{\Phi}^T \boldsymbol{y}_k, \boldsymbol{f}_\tau^\rho \rangle|.$$

- Efficiently implemented as convolution

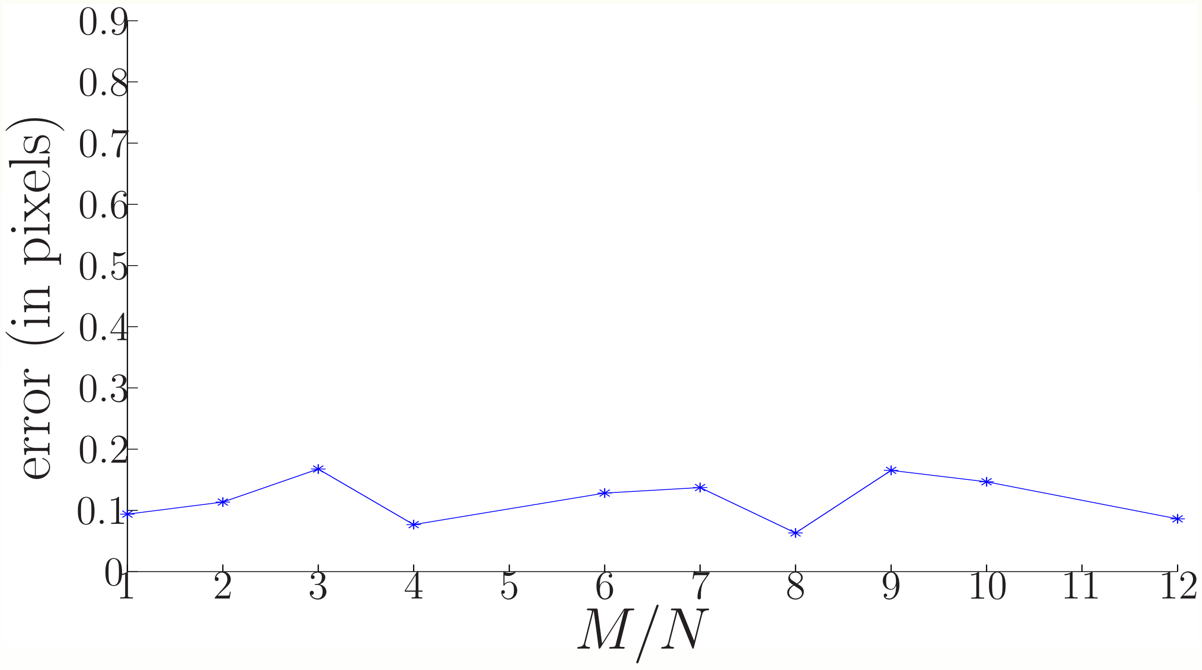
- Large enough M
  - achieves maximum at same location as in signal domain

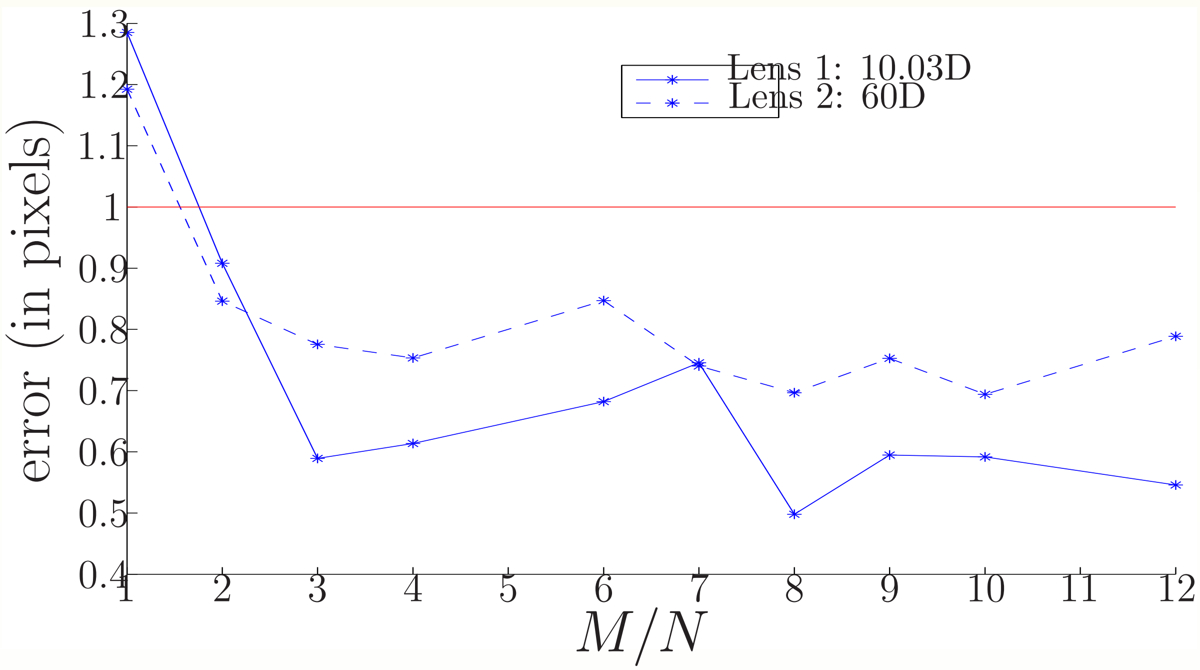
- Translation estimation = 1 convolution + 1 least squares

Experimental results

- Synthetic
  - randomly generated Gaussian patterns
  - ground truth known
- Experimental
  - deflectometric data of two lenses
  - reference translations: obtain using full reconstruction

- Error: absolute pixel error between estimate and reference





Summary

- A signal model and method to estimate continuous parameters from compressive measurements

- Possible interesting direction:
  - Learn the function $f_c(t)$
  - others?

- References

1) Ekanadham, C., Tranchina, D., & Simoncelli, E. P. (2011). Recovery of Sparse Translation-Invariant Signals With Continuous Basis Pursuit. *IEEE TSP*, *59*(10), 4735–4744.

2) Fyhn, K., Duarte, M. F., & Jensen, S. H. (2013, May 15). Compressive Parameter Estimation for Sparse Translation-Invariant Signals Using Polar Interpolation. arXiv.org.

3) Jacques, L., & De Vleeschouwer, C. (2008). A Geometrical Study of Matching Pursuit Parametrization. Signal Processing, IEEE Transactions on, 56(7), 2835–2848.

4) Davenport, M. A., Duarte, M. F., Wakin, M., Laska, J., Takhar, D., Kelly, K., & Baraniuk, R. (2007). The smashed filter for compressive classification and target recognition. Proc. SPIE Computational Imaging V, 6498.