Solving imaging problems with Graph Cuts

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Markov Random Field: definitions

MRF

X defined over a lattice $\nu = \{1, 2, ..., n\}$ with a neighbourhood system N_i. Each random variable $Xi \in \mathbf{X}$ is associated with a lattice point $i \in \nu$ and takes a value from the label set $\mathcal{L} = \{l_1, l_2, ..., l_k\}$.

Clique

A *clique* c is a set of random variable $\mathbf{X}_c$ which are conditionally dependant one to each other. The *order* of a clique is defined as the number of variable in the clique minus one.

Labeling

Any possible assignment of labels to the random variables is called a labelling. It is denoted by the vector **x**, and takes values from the set $\mathbf{L} = \mathcal{L}^n$.

Maximum a Posteriori labelling

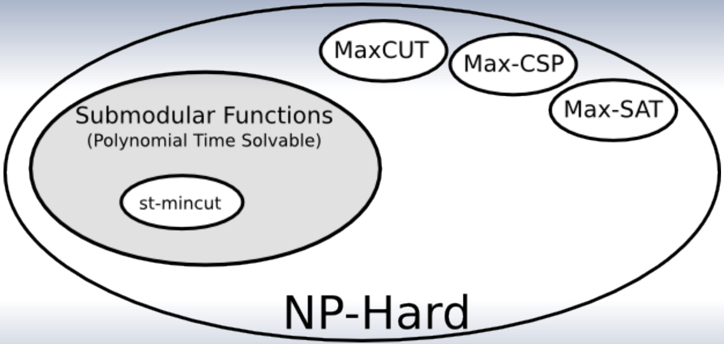
Gibbs Energy

$E(\mathbf{x}) = \sum_c \psi_c(\mathbf{x}_c)$

MAP labelling

The maximum a posteriori (MAP) labelling $\mathbf{x}^*$ of a random field is defined as $\mathbf{x}^* = \arg\max_{\mathbf{x}\in\mathbf{L}} -E(\mathbf{x})$ and can be found by minimizing the energy function E.

Discrete Energy Minimization (1)



Discrete Energy Minimization (2)

Submodular function

Consider the set $N = \{1, 2, \dots, n\}$. A set function $f_s : 2^N \to \mathbb{R}$ is said to be submodular iff $\forall A, B \subseteq N$ the function satisfies:

$$f_s(A) + f_s(B) \geq f_s(A \cap B) + f_s(A \cup B)$$

Example if $N = \{1, 2\}$, the variables X_1 and X_2 are binary, the above condition becomes:

$$f_b(0,1) + f_b(1,0) \geq f_b(1,1) + f_b(0,0)$$

We will suppose for now that we have binary variables.

Minimizing Submodular Functions using Graph Cuts (1)

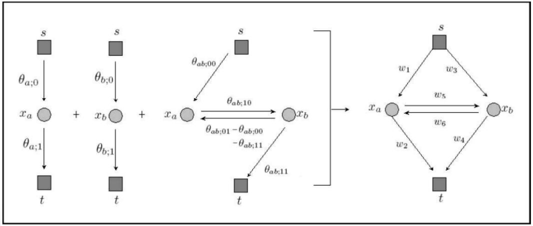
Any second order function of binary variables could be wrote as:

$$E(\mathbf{x}|\theta) = \theta_{const} + \sum_{v \in \nu, i \in \mathcal{L}} \theta_{v;i}\delta_i(x_v) + \sum_{(u,v) \in \nu, (j,k) \in \mathcal{L}^2} \theta_{uv;jk}\delta_k(x_v)\delta_j(x_v)$$

- $\theta_{v,i}$: the penalty for assigning the label i to latent variable x_v
- $\theta_{uv,jk}$: the penalty for assigning labels j and k to variables x_u and x_v
- $\delta_j(x_u)$: indicator function, equals to 1 if the $x_u = j$
- $E(\mathbf{x}, \theta)$ is submodular if $\theta_{uv,jk} \geq 0 \forall (u, v) \in \nu, \forall (j, k) \in \mathcal{L}^2$

Minimizing Submodular Functions using Graph Cuts (2)

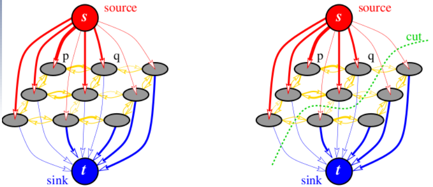
$$E(\mathbf{x}|\theta) = \theta_{const} + \sum_{v \in \nu, i \in \mathcal{L}} \theta_{v;i}\delta_i(x_v) + \sum_{(u,v) \in \nu, (j,k) \in \mathcal{L}^2} \theta_{uv;jk}\delta_k(x_v)\delta_j(x_v)$$



[1]

P. Kohli.
Minimizing dynamic and higher order energy functions using graph cuts.
PhD thesis, Oxford University, 2007.

Minimizing Submodular Functions using Graph Cuts (3)



- the cost of an *st-cut* is equal to the energy $E(\mathbf{x}|\theta)$ of its corresponding configuration $\mathbf{x}$,
- **the minimal cost could be computed in $O(mn^2)$ if the function is submodular, n & m are the number of nodes and edges in $\mathcal{G}$**, [1].

Yuri Boykov and Vladimir Kolmogorov.
An experimental comparison of min-cut/max-flow algorithms for energy minimization in vision.
IEEE Trans. Pattern Anal. Mach. Intell., 26(9):1124–1137, 2004.

Graph Cuts and segmentation

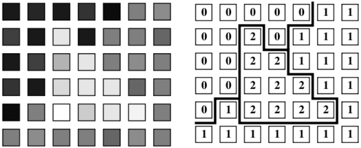
How can Graph Cuts be applied to segmentation?

- defining the parameters,
- some models,
- user interaction,
- higher order potentials

Graph Cuts and segmentation: set of labels

[1]

$$\mathcal{L} = \{0, 1\} = \{\textit{foreground}, \textit{background}\}$$

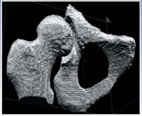


$$\mathcal{L} = \{0, 1, 2\}$$

- The size of the set of labels has no impact on the submodular characteristic.

J. Malcolm, Y. Rathi, and A. Tannenbaum.
A graph cut approach to image segmentation in tensor space.
pages 1–8, 2007.

Graph Cuts and segmentation: unary costs

log-likelihood: $\theta_{v,i} = -\ln Pr(I_v | foreground)$  [1]

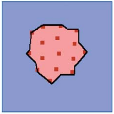
- The unary potential has no impact on the submodular characteristic.
- Any unary potential is submodular.

Yuri Boykov and Gareth Funka-Lea.
Graph Cuts and Efficient N-D Image Segmentation.
Int. J. Comput. Vision, 70(2):109–131, 2006.

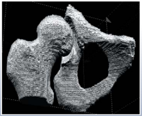
Graph Cuts and segmentation: pairwise costs

- This term must be submodular to benefit from the polynomial time solvable characteristic.
- It introduces some kind of smoothness in the model.
- Many models are known to submodular

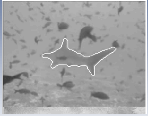
Gaussian model



Potts model



Graph Cuts and segmentation: pairwise costs

shape prior  [1]

active contour  [2]

J. Malcolm, Y. Rathi, and A. Tannenbaum.
Graph cut segmentation with nonlinear shape priors.
Image Processing, 2007. ICIP 2007. IEEE International Conference on, 4:IV –365–IV –368, 16 2007-Oct. 19 2007.

N. Xu, N. Ahuja, and R. Bansal.
Object segmentation using graph cuts based active contours.
Computer Vision and Image Understanding, 107(3):210–224, 2007.

Graph Cuts and segmentation: user interactions

User interactions are very easy to incorporate into Graph Cuts model.

- get prior knowledges (unary costs),
- fix the labels to certain pixels
- easily correct a solution by changing local costs,
- re-use the previous flow to faster compute a new solution.


[1]

Yuri Boykov and Gareth Funka-Lea.
Graph Cuts and Efficient N-D Image Segmentation.
Int. J. Comput. Vision, 70(2):109–131, 2006.

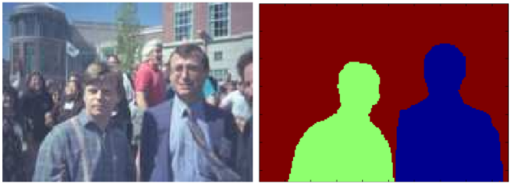
Graph Cuts and segmentation: limitations and higher order clique

Using higher order clique?

- all previous models are first order clique,
- general submodular conditions are known for second order clique only,
- very few general high order models are known to be submodular
- what is the set of submodular energy functions?
    - Still an unsolved question.
    - NP=P?

Graph Cuts and segmentation: limitations and higher order clique

General Potts model

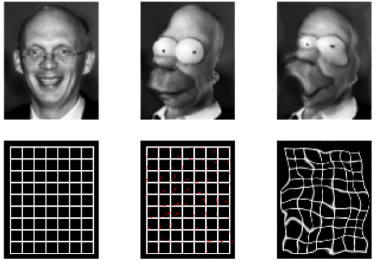


[1]

Pushmeet Kohli, L'Ubor Ladický, and Philip H. Torr.
Robust higher order potentials for enforcing label consistency.
Int. J. Comput. Vision, 82(3):302–324, 2009.

Graph Cuts and Registration

How can Graph Cuts be applied to segmentation?

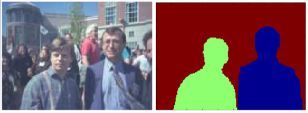


Graph Cuts and Registration

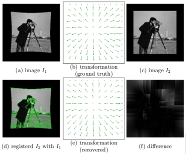
- labe sets: $\mathcal{L} = \{disp.\}$, all the displacements
  - could be very large.
- Unary cost
  - the penalty to move a pixel u with a displacement d_u,
  - always submodular, every metric could be used.
- Pairwise term
  - smoothing the displacement field,
  - submodularity must be check.

Graph Cuts and Registration

Atlas registration


[2]

Landmarks and non-rigid registration



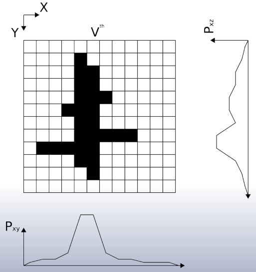
[1]

Herve Lombaert, Yiyong Sun, and Farida Cheriet.
Landmark-based non-rigid registration via graph cuts.
In *Proc. of the 4th International Conference, ICIAR 2007*, volume 4633, pages 166–175, 2007.

F. van der Lijn, T. den Heijer, M. Breteler, and W.J. Niessen.
Hippocampus segmentation in MR images using atlas registration, voxel classification, and graph cuts.
NeuroImage, 43(4):708–720, 2008.

Graph Cuts and Discrete Tomography

How can Graph Cuts be applied to Discrete Tomography?



Graph Cuts and Discrete Tomography

- What's happen when we deal with such kind of problem?
  - Much higher order clique
  - No more submodular (*in general*)
- Solutions? Still a very challenging problem.

Other applications

- image denoising,
- surface reconstruction,
- . . .
- any problem which could be formulated as submodular lablization problem.